The kernel of a homogeneous derivation of $k\,[X, Y]$

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Abstract

Let k be an algebraically closed field of characteristic zero, δ a homogeneous derivation of $k\,[X, Y]$. We give a criterion for the existence of a non-constant polynomial F such that $\delta\ (F) = 0$.

Keywords

homogeneous derivation, ring of constants.

Let k be an algebraically closed field of characteristic zero, $R=k\,[X, Y]$ a polynomial ring in two variables. Let $\delta = f\partial/\partial X + g\partial Y \neq 0$ be a k-derivation of R $(f, g \in R)$. We write R^{δ} for the kernel of δ. It is known that R^{δ} is a subring of R and that $R^{\delta}=k\,[F]$ for some $F \in R$ (cf. [2] Thoerem 2.8). In this note we treat the case where δ is homogeneous, that is, f and g are both homogeneous polynomials and deg (f) =deg (g). We remark that any homogeneous irreducible polynomial in R is a linear form in X and Y as k is algebraically closed.

To consider R^{δ}, we may assume that f and g are relatively prime. First we assume that $Xg-Yf=0$. Then we have $f=X$ and $g=Y$. Hence $R^{\delta}=k$.

From now on, we assume that $Xg-Yf \neq 0$.

Lemma 1 *Let a and b be homogeneous polynomials in R. Assume that a is sqare-free and that* deg (a) = deg (b) +1. Let $a=\Pi_{i=1}^{d}\,(\alpha_i X+\beta_i Y)$ *be a factorization into irreducibles where* d=deg (a). We set $a_i = \Pi_{j\neq i}\,(\alpha_j X+\beta_j Y)$ for $i = 1, \ldots,$ d. *Then we have*

$$b=\sum_{i=1}^{d} b(\beta_i - \alpha_i)\ \alpha_i\,(\beta_i - \alpha_i)^{-1}\alpha_i$$

Proof. Put $c = b - \Sigma_{i=1}^{d}\, b(\beta_i - \alpha_i)\, a_i\ (\beta_i - \alpha_i)^{-1}\, a_i$. Then c is a homogeneous polynomial of degree $d-1$. Since $c\,(\beta_i - \alpha_i) = 0$ for $i=1, \ldots,$ d, we have $c=0$.

Let $Xg-Yf = \Pi_{i=1}^{r}\ (\alpha_i X_i + \beta_i Y_i)$ be a factorization into irreducibles. We set $\Delta_i = \Pi_{j\neq i}\,(\alpha_j \beta_i - \alpha_i \beta_j)$ for $i=1, \ldots, r$.

Theorem 2 *Let the notaion and the assumption be as above. Then there exists a non-constant* *polynomial $F \in R$ shch that $\delta(F)=0$ if and only if the following two conditions are satisfied* :

(i) *$Xg-Yf$ is square-free* ;
(ii) *there exist integers $m_1, \ldots, m_r \geq 1$ and a constant $\gamma \in k$ such that*

$$f(\beta_i-\alpha_i)=-\gamma m_i \beta_i \Delta_i \text{ and } g(\beta_i-\alpha_i)=\gamma m_i \alpha_i \Delta_i \text{ for } i=1, \ldots, r.$$

Proof. Assume that the conditions (i) and (ii) are satisfied. We put $F=\Pi_{i=1}^{r}(\alpha_i X+\beta_i Y)^{m_i}$. Then, by LEMMA 1, we have

$$f=\sum_{i=1}^{r} f(\beta_i,-\alpha_i)\Delta_i^{-1}\prod_{j\neq i}(\alpha_j X+\beta_j Y)$$

$$=-\gamma\sum_{i=1}^{r} m_i \beta_i \prod_{j\neq i}(\alpha_j X+\beta_j Y),$$

and hence

$$\partial F/\partial Y=-\gamma^{-1}\prod_{i=1}^{r}(\alpha_i X+\beta_i Y)^{m_i-1} f.$$

Similarly we have

$$\partial F/\partial X=\gamma^{-1}\prod_{i=1}^{r}(\alpha_i X+\beta_i Y)^{m_i-1} g.$$

Thus we have $\delta(F)=0$.

Conversely, assume that there exists a polynomial $F\in R\setminus k$ such that $\delta(F)=0$. Since δ is homogenous we may assume that F is a homogeneous polynomial. Then there exists a homogeneous polynomial $h\in R$ such rhat $\partial F/\partial X=-fh$ and $\partial F/\partial Y=gh$ as f and g are relatively prime. Since $\deg(F)F=X\partial F/\partial X+Y\partial F/\partial Y=(Xg-Yf)h$, h is a greatest common divisor of F, $\partial F/\partial X$ and $\partial F/\partial Y$. Let $F=\Pi_{i=1}^{s} p_s^{a_i+1}$ be a factorization to distinct irreducibles ($a_i>0$). Then $\Pi_{i=1}^{s} p_s^{a_s}$ is a greatest common divisor of F, $\partial F/\partial X$ and $\partial F/\partial Y$ (*cf. e.g.* [1] Chapter 4, §2, proposition 12). Hence we have $h=c\Pi_{i=1}^{s} p_s^{a_i}$ for some $c\in k$ and $(Xg-Yf)=\deg(F)c^{-1}\Pi_{i=1}^{s} p_s$. Thus $Xg-Yf$ is square-free. Moreover we may assume that $F=\Pi_{i=1}^{r}(\alpha_i X_i+\beta_i Y_i)^{a_i+1}$. Thus we have

$$\partial F/\partial Y=\sum_{i=1}^{r}(a^i+1)\ \alpha_i\prod_{i=1}^{r}(\alpha_j X+\beta_j Y)^{a_j}\prod_{j\neq i}(\alpha_j X+\beta_j Y),$$

and hence

$$g=\gamma\sum_{i=1}^{n}(a_i+1)\ \alpha_i\prod_{j\neq i}(\alpha_j X+\beta_j Y) \text{ where } \gamma=c^{-1};$$

Similarly we have

$$f=-\gamma \sum_{i=1}^{n} (a_i+1)\ \beta_i \prod_{j\neq i} (\alpha_j X+\beta_j Y),$$

which shows the condition (ii).

Refernces

[1] Cox, D., Little, J. and O'Shea, D., Ideals, Varieties, and Algorithms, Springer, 1992, New York.

[2] Nagata, M. and Nowicki, A., Rings of constants for k-derivations in k $\{x_1,\dots, x_n\}$, J. Math. Kyoto Univ., **28** (1988), 11-118.