Weak Fano 3-folds with a quadric bundle structure

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ABSTRACT. The aim of this article is a classification of weak Fano 3-folds with a quadric bundle structure, under the assumption that the 3-folds have only finite (-2)-curves and their Picard number is two. The 3-folds are classified into only 9 types and the structure of a 3-fold belonging to each type can be determined in detail.

Received Apr. 28, 1997

Key words : Fano 3-fold, quadric bundle

§. 0. Introduction

A smooth 3-fold is called a *Fano 3-fold* if its anti-canonical divisor is ample. Fano 3-folds have been classified by Fano, Iskovskih, Shokurov, Mori, and Mukai. Let us consider 3-folds with weaker conditions. Since nef & big divisors are in a boundary of an ample cone, a weak Fano 3-fold is defined as follows in this paper.

Definition. A smooth 3-fold is called a *weak Fano 3-fold* if its anti-canonical divisor is nef & big.

M.Reid [R2] has already introduced weak Fano 3-folds which may have canonical singularities, but here we confine ourselves weak Fano 3-folds having no singularities.

A projective morphism f: $V \to C$ from a 3-fold V onto a smooth curve C is called a *quadric bundle* if its generic fiber V_η is a quadric surface. This is closely related to del Pezzo fibrations which arise out of extremal ray contractions of 3-folds. The contraction of an extremal ray is a morphism of relative Picard number one, so that the quadric bundles investigated here are confined to having relative Picard number one.

In the minimal model theory, the phenomenon called *flop* appears and plays an important role. In the case of dimension three, the flop arises from a (-2)-*curve*, which is a curve $C \cong \mathbb{P}^1$ on a 3-fold whose normal bundle is isomorphic to either $\mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$ or $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$ ([R1]). In general, these curves are contracted to cDV points. While an isolated (-2)-curve is contracted to a terminal singularity, a non-isolated one is contracted to a non-terminal singularity and has a family including infinite (-2)-curves. Since terminal singularity is a slight extension of non-singular point, it is natural to consider weak Fano 3-folds having finite (-2)-curves.

Putting the above conditions together, we investigate weak Fano 3-folds which have a quadric bundle structure of relative Picard number one and only finite (-2)-curves. Then we can show that the 3-folds are classified into finite types. This article provides the classification for them.

The next section discribes the main result. In §.2, we recall the general result on quadric bundles and derive three inequalities playing important role in this paper. The final section proves the main result.

The author is grateful to Prof.S.Mukai, Prof.S.Mori, and Mr. T. Maeda for helpful suggestion.

§. 1. Main result

(1.0) In this section, we recall some definitions and state the main theorem in this paper.

(1.1) **Definition**. A projective morphism $\varphi : V \to C$ from a 3-fold V to a smooth curve C is called a *quadric bundle* over C if its generic fiber is isomorphic to a quadric surface. A 3-fold V has a *quadric bundle structure* if there exists a projective morphism $\varphi: V \to C$ which is a quadric bundle.

(1.2) **Definition**. A smooth projective 3-fold V is a *weak Fano 3-fold* if its anti-canonical divisor $-K_V$ is nef & big, and V is a *Fano 3-fold* if $-K_V$ is ample.

The notion of weak Fano 3-folds was introduced by M.Reid [R2] to 3-folds having canonical singularities. However, our weak Fano 3-folds have no singularities in this paper.

(1.3) We consider a smooth projective 3-fold V which is a weak Fano 3-fold with only finite (-2)-curves, i.e., which has the properties that its multi-anti-canonical linear system $|-mK_V|$ is free, that the corresponding morphism contracts only finite curves, and that the image of this morphism is a 3-fold with only terminal singularities.

The aim of this article is to classify weak Fano 3-folds with a quadric bundle structure and with only finite (-2)-curves. The result is:

(1.4) **Theorem**. *Every weak Fano 3-fold V with only finite (-2)-curves, such that V has a quadric bundle structure with relative Picard number 1, is isomorphic to a general member of a linear system Λ on a $\mathbb{P}^3$-bundle $X = \mathbb{P}(E)$ over a projective line $\mathbb{P}^1$, where E and Λ are one of the following.*

(1.4.1) $\qquad E = O^{\oplus 3} \oplus O(1), \qquad \Lambda = |2H|$

(1.4.2) $E = O \oplus O(1)^{\oplus 3}$, $\Lambda = |2H-F|$
(1.4.3) $E = O^{\oplus 4}$, $\Lambda = |2H+F|$
(1.4.4) $E = O^{\oplus 2} \oplus O(1)^{\oplus 2}$, $\Lambda = |2H|$
(1.4.5) $E = O^{\oplus 3} \oplus O(1)$, $\Lambda = |2H+F|$
(1.4.6) $E = O \oplus O(1)^{\oplus 3}$, $\Lambda = |2H|$
(1.4.7) $E = O^{\oplus 4}$, $\Lambda = |2H+2F|$
(1.4.8) $E = O^{\oplus 2} \oplus O(1)^{\oplus 2}$, $\Lambda = |2H+F|$
(1.4.9) $E = O \oplus O(1)^{\oplus 3}$, $\Lambda = |2H+F|$

Here H is the tautological line bundle and F is a fiber of the $\mathbf{P}^3$-bundle $\pi : X \to \mathbf{P}^1$ containing $\varphi : V \to \mathbf{P}^1$.

(1.5) **Supplement**. In the above list, each 3-fold V has the following properties.

(1.5.1) $(-K_V)^3 = 40$, V is a Fano 3-fold which is a blowing-up of a quadric 3-fold $\mathbf{Q}^3$ along a smooth conic ${}_0C_2$.

(1.5.2) $(-K_V)^3 = 40$, V has only one (-2)-curve s_0 as a section, and the 3-fold V' obtained by $(-F_V)$-flop of V has a $\mathbf{P}^1$-bundle structure over $\mathbf{P}^2$ which contains a strict transform s_0' of s_0 as a subsection, where F_V is a fiber of φ: $V \to \mathbf{P}^1$.

(1.5.3) $(-K_V)^3 = 32$, V is a Fano 3-fold which is a blowing-up of $\mathbf{P}^3$ along a quartic elliptic curve ${}_1C_4$.

(1.5.4) $(-K_V)^3 = 32$, V has just two (-2)-curves as sections, and the 3-fold V' obtained by $(-F_V)$-flop of V has a quadric bundle structure again.

(1.5.5) $(-K_V)^3 = 24$, V has just four (-2)-curves as sections, and the 3-fold V' obtained by $(-F_V)$-flop of V is a blowing-up at a point of B_4, where B_4 is a Fano 3-fold of index 2, a complete intersection of two quadrics in $\mathbf{P}^5$.

(1.5.6) $(-K_V)^3 = 24$, V is a Fano 3-fold and has a conic bundle structure over $\mathbf{P}^2$ with the discriminant locus Δ of degree 4.

(1.5.7) $(-K_V)^3 = 16$, V has just eight (-2)-curves as sections, and the 3-fold V' obtained by $(-F_V)$-flop of V has a quadric bundle structure again.

(1.5.8) $(-K_V)^3 = 16$, V has only one (-2)-curve C as a bisection, and the 3-fold V' obtained by $(-F_V)$-flop of V has a structure of a del Pezzo fibration of degree 4 over $\mathbf{P}^1$ which contains a strict transform C' of C as a section.

(1.5.9) $(-K_V)^3 = 8$, V has just eighteen (-2)-curves C_1, C_2, $s_1, \cdots, s_{16}$, two of which C_1 and C_2 are bisections and the others are sections of the fibration, and the 3-fold V' obtained by $(-F_V)$-flop of V has a quadric bundle structure again.

§. 2. Preliminaries

(2.0) From the general theory of quadric bundles (e.g. A.Beauville [B]), we can represent any quadric bundle as a hypersurface in a $\mathbb{P}^3$-bundle.

Theorem. *Any quadric surface bundle* V *over* $\mathbb{P}^1$ *is embedded in* $X = \mathbb{P}(E)$ *as a hypersurface defined by a global section of* $S^2(E) \otimes O(k)$ *for some integer* k*, where* $E = O \oplus O(a_1) \oplus O(a_2) \oplus O(a_3)$ *and* $S^2(E)$ *is a symmetric tensor product of* E*.*

(2.1) From now on we fix the condition discribed in the above theorem with the following notation. Let $E = O \oplus O(a_1) \oplus O(a_2) \oplus O(a_3)$, $0 \leqq a_1 \leqq a_2 \leqq a_3$, be a locally free sheaf of rank 4 on $\mathbb{P}^1$, and $\pi : \mathbb{P}(E) = X \to \mathbb{P}^1$ the corresponding $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. Let H be the tautological line bundle and let F be a fiber of $\pi : X \to \mathbb{P}^1$.

Then the set of numerical equivalence classes of divisors and 1-cycles on X are

$$N^1(X) = \mathbb{R}[H] \oplus \mathbb{R}[F], \quad N_1(X) = \mathbb{R}[l] \oplus \mathbb{R}[s_0],$$

respectively. Here l is a line in a fiber and s_0 is the minimal section associated to the exact sequence $E \to O \to 0$; their classes form the dual basis of $[H]$ and $[F]$ with respect to intersection number. The closure $\overline{NE}(X)$ of a convex cone generated by numerical classes of effective 1-cycles of X (Mori cone of X) is

$$\overline{NE}(X) = \mathbb{R}_+[l] + \mathbb{R}_+[s_0] \subset N_1(X).$$

Here $\mathbb{R}_+$ denotes the set of non-negative real numbers.

Let

$$\begin{array}{ccc} V & \xrightarrow{\psi} & \mathbb{P}(E) = X \\ {\scriptstyle\varphi}\downarrow & \swarrow{\scriptstyle\pi} & \\ \mathbb{P}^1 & & \end{array}$$

be the diagram stated in (2.0): ψ is an embedding. In addition, we assume that the Picard number $\rho(V)$ of V equals 2. Then, the restriction map $\psi^*: N^1(X) \to N^1(V)$ is an isomorphism and so is ψ_*: $N_1(V) \to N_1(X)$. Under the isomorphism ψ_*, $N_1(V)$ is identified with $N_1(X)$, a Mori cone $\overline{NE}(V)$ of V maps into $\overline{NE}(X)$ but not surjectively in general, and the edges of $\overline{NE}(V)$ are generated by $[l]$ and $[C_V]$ for some rational curve $C_V \subset V$ with $(\psi^* H \cdot C_V) \geqq 0$:

$$\begin{array}{ccc} N_1(V) & \xrightarrow[\psi_*]{\sim} & N_1(X) \\ \cup & & \cup \\ \mathbb{R}_+[l]+\mathbb{R}_+[C_V]=\overline{NE}(V) & \longrightarrow & \overline{NE}(X)=\mathbb{R}_+[l]+\mathbb{R}_+[s_0] \end{array}$$

(2.2) We denote $H_V = \psi^* H$ and $F_V = \psi^* F$. For the sake of convenience, we set

$\mu(Z) = \frac{(H\cdot Z)}{(F\cdot Z)} \left(= \frac{(H_V\cdot Z)}{(F_V\cdot Z)} \right)$ for each 1-cycle Z on $V \subset X$, and call it a *slope* of Z. Moreover we set $\mu(R) = \mu(Z)$ for any ray $R = \mathbb{R}_+ [Z] \subset N_1(V)$, and call it a *slope* of R.

If V has finite (-2)-curves, there is a flop χ: $V \cdots\rightarrow V'$ corresponding to an edge R_0 of $\overline{NE}(V)$. We have a canonical isomorphism χ^*: $\mathrm{Pic}(V') \rightarrow \mathrm{Pic}(V)$, and we can identify $N^1(V')$ with $N^1(V)$ by χ^*: $N^1(V') \rightarrow N^1(V)$; put $H_{V'} = \chi^* H_V$, $F_{V'} = \chi^* F_V$. As stated above, we can consider a slope $\mu(Z') = \frac{(H_{V'}\cdot Z')}{(F_{V'}\cdot Z')}$ of 1-cycle Z' on V'. If Z' is a strict transform of $Z \subset V$, we have the following expression of the value $\mu(Z')$.

Let $C_1, \cdots, C_k$ be all the curves in V of slope $\mu(R_0)$, and put a $= (H_V\cdot Z)$, b $= (F_V\cdot Z)$, $\alpha_i = (H_V\cdot C_i)$, $\beta_i = (F_V\cdot C_i)$, $n_i = \#\{Z \cap C_i\}$ for each i. Then

(2.2.1) $$\mu(Z') = \frac{a + \Sigma_{i=1}^k n_i\alpha_i}{b + \Sigma_{i=1}^k n_i\beta_i}.$$

This formula is useful in finding an extremal ray of $\overline{NE}(V')$.

(2.3) Consider a sub-$\mathbb{P}^1$-bundle T in X associated to the exact sequence
$$E = O \oplus O(a_1) \oplus O(a_2) \oplus O(a_3) \longrightarrow F = O \oplus O(a_i) \longrightarrow 0,$$
of sheaves and denote the pull-back $\psi^* T$ in V by D. If D is an effective curve, then, the first inequality

(2.3.1) $$(D\cdot -K_V) \geqq 0$$

is obtained from the nefness of $-K_V$. Since H_V is nef, the second inequality

(2.3.2) $$(D\cdot H_V) \geqq 0$$

holds. The third inequality

(2.3.3) $$(-K_V)^3 > 0$$

follows from the bigness of $-K_V$.

(2.4) The following lemma assures that D in (2.3) can be used as a substitute for the edge curve C_V of the Mori cone $\overline{NE}(V)$.

Lemma. *Let* $0 \leqq a_1 = \cdots = a_k (=a) < a_{k+1} \leqq \cdots \leqq a_3$ *be a sequence of integers and let* $E = O \oplus O(a)^{\oplus k} \oplus O(a_{k+1}) \oplus \cdots \oplus O(a_3)$ *be a locally free sheaf on* $\mathbb{P}^1$. *Assume that* C *is a rational curve in* X *with* $\mu(C) \leqq a$. *Then, the curve* C *is on a ruled surface* $S_\lambda = \mathbb{P}(O \oplus O(a))$ *determined by* $\lambda \in \mathbb{P}^{k-1}$.

Proof. Let D be a normalization of C and let μ: $D \rightarrow X$ and $\sigma = \pi \circ \mu$: $D \rightarrow \mathbb{P}^1$ be natural morphisms. Denote $(F\cdot C)$ and $(H\cdot C)$ respectively by d and b. Then, we have $\deg\sigma = d$ and $\sigma^* E \rightarrow O_C(b) \rightarrow 0$. Since $\sigma^* O(1) = O_C(d)$, we have the exact sequence

(2.4.1) $$O_C \oplus O_C(da_1) \oplus O(da_2) \oplus O_C(da_3) \longrightarrow O_C(b) \longrightarrow 0.$$

If $0 \leqq b < da$, then the sequence (2.4.1) factors through O_C and the curve C is a multiple of the unique minimal section $s_0 = \mathbb{P}(O) \subset X$ associated to $E \to O \to 0$. In this case, the assertion is trivial.

Now we assume that $0 \leqq b = da$. Then, the sequence (2.4.1) factors through $O_C \oplus O_C(b)^{\oplus k}$. We can choose a basis such that $O_C \oplus O_C(b) \to O_C(b) \to 0$ in the subsheaf $O(b)^{\oplus k}$ of a direct sum of copies, and, hence, the morphism μ factors through $S_\lambda = \mathbb{P}(O \oplus O(a))$, where $\lambda \in \mathbb{P}^{k-1}$ indicates the choice of the basis. □

(2.5) We end off this section by recalling the Lefschetz theorem on the Picard group with its corollaries.

(2.5.1) **Theorem.** ([H] p.178) *Let X be a complete non-singular variety, and let Y be a closed subscheme. Assume*

(i) Leff(X,Y),

(ii) *Y meets every effective divisor on X, and*

(iii) $H^i(Y, I^n/I^{n+1}) = 0$ *for* $i = 1, 2$, *and all* $n \geqq 1$, *where I is the sheaf of ideals of Y.*

Then the natural map Pic$(X) \to$ Pic(Y) *is an isomorphism.*

Here Leff(X,Y) is the effective Lefschetz condition, this holds as in the following case:

(2.5.2) **Theorem.** ([H] p.172) *Let X be a non-singular subvariety of* $\mathbb{P}^N_k$, *and let Y be a closed subscheme of codimension r, which is a complete intersection (i.e.,* $Y = X \cap H_1 \cap \cdots \cap H_r$ *where the* H_i *are hypersurfaces in* $\mathbb{P}^N_k$*). Assume that* dim $Y \geqq 2$. *Then we have* Leff(X,Y).

Now we derive some corollaries of (2.5.1) used in the following sections.

(2.5.3) **Corollary**. *Let X be a non-singular projective variety, and let Y be an ample effective divisor on X. Assume that* dim $Y \geqq 3$. *Then the natural map* Pic$(X) \to$ Pic(Y) *is an isomorphism.*

Proof. This is obvious from (2.5.2) and (2.5.1).

(2.5.4) **Corollary**. *Let* X' *be a non-singular projective variety, let* $Y' = \bigcap_i D'_i$ *be a complete intersection of n ample effective divisors* $D'_i \subset X'$, *and let* Z' *be a non-singular subvariety of* X'. *Let* $f: X \to X'$ *be a blowing-up of* X' *along* Z', *and let* $Y \subset X$ *be a strict transform of* Y'. *Assume that* $Z' \cap Y'$ *is non-singular and irreducible, and that* dim $Y \geqq 3$. *Then the natural map* Pic$(X) \to$ Pic(Y) *is an isomorphism.*

Proof. We have Pic$(X') \cong$ Pic(Y') by (2.5.3), and Pic$(X) \cong f^*(\text{Pic}(X')) \oplus \mathbb{Z}$.

From the assumptions in the corollary, we have $\mathrm{Pic}(Y) \cong f^*(Pic(Y')) \oplus \mathbb{Z}$.

Applying the above corollary to the morphism f: $\mathbb{P}(O^{\oplus n} \oplus O(1)) \to \mathbb{P}^{n+1}$ defined by the linear system $|H|$, we have the following two corollaries.

(2.5.5) **Corollary.** *Let X be a $\mathbb{P}^n$-bundle $\mathbb{P}(O^{\oplus n} \oplus O(1))$ over $\mathbb{P}^1$, and let Y be a complete intersection of general members of linear systems $|m_iH|$ on X for positive integers m_i. Assume that* $\dim Y \geqq 3$. *Then* $\mathrm{Pic}(Y) \cong \mathbb{Z}^{\oplus 2}$.

(2.5.6) **Corollary.** *Let X be a $\mathbb{P}^n$-bundle $\mathbb{P}(O^{\oplus n-1} \oplus O(1)^{\oplus 2})$ over $\mathbb{P}^1$, and let Y be a complete intersection of general members of linear systems $|m_iH|$ on X for positive integers m_i. Assume that* $\dim Y \geqq 3$. *Then* $\mathrm{Pic}(Y) \cong \mathbb{Z}^{\oplus 2}$.

Proof. For the morphism f: $X \to \mathbb{P}^{n+2}$ defined by the linear system $|H|$, it is obtained that the $\mathbb{P}^{n+1}$-bundle $W = \mathbb{P}(O^{\oplus n+1} \oplus O(1))$ contains X as an ample effective divisor, and that f is a restriction of the morphism g: W $\to \mathbb{P}^{n+2}$ defined by $|H_W|$ associated to the tautological line bundle of $W \to \mathbb{P}^1$. There is a complete intersection V' of hypersurfaces in $\mathbb{P}^{n+2}$ such that $Y = V \cap X$, where V is a pull-back of V'. We have $\mathrm{Pic}(V) \cong \mathrm{Pic}(Y)$ by (2.5.3), and $\mathrm{Pic}(V) \cong \mathbb{Z}^{\oplus 2}$ by generality of V'.

(2.5.7) **Corollary.** *Let X be a $\mathbb{P}^n$-bundle $\mathbb{P}(O \oplus O(1)^{\oplus n})$ over $\mathbb{P}^1$, and let Y be a general member of a linear system $|mH|$ on X for a positive integer m, where H is the tautological line bundle of $X \to \mathbb{P}^1$. Assume that* $\dim Y \geqq 3$. *Then* $\mathrm{Pic}(Y) \cong \mathbb{Z}^{\oplus 2}$.

Proof. Modifying the argument of proof of (2.5.2) (described in [H]) slightly, we obtain $\mathrm{Leff}(X, Y)$ in this case too. The conditions (ii) and (iii) of (2.5.1) are checked immediately. Thus we have $\mathrm{Pic}(Y) \cong \mathbb{Z}^{\oplus 2}$ since $\mathrm{Pic}(X) \cong \mathbb{Z}^{\oplus 2}$.

§. 3. Quadric bundles

(3.0) In this section, we will derive the main results (1.4) and (1.5). Let V be a quadric bundle in $X = \mathbb{P}(O \oplus O(a_1) \oplus O(a_2) \oplus O(a_3))$, $0 \leqq a_1 \leqq a_2 \leqq a_3$, such that $V \in |2H + kF|$ (see (2.0) and (2.1)):

$$\begin{array}{ccl} V & \xrightarrow{\psi} & X = \mathbb{P}(O \oplus O(a_1) \oplus O(a_2) \oplus O(a_3)) \\ \varphi \downarrow & \swarrow \pi & \qquad V \in |2H + kF| \\ \mathbb{P}^1 & & \end{array}$$

Assume that V is a weak Fano 3-fold with only finite (-2)-curves and with Picard number $\rho(V) = 2$.

(3.1) Under this situation (3.0), the anti-canonical divisor of V is

$$-K_V \sim 2H_V - (\sum_{i=1}^{3} a_i + k - 2) F_V,$$

the intersection numbers are

$$(H_V^3) = 2\sum_{i=1}^{3} a_i + k,\ (H_V^2 \cdot F_V) = 2, \text{ and } (l \cdot H_V) = (s_0 \cdot F_V) = 1,$$

and the curve D described in (2.3) is

$$D \equiv (H_V - a_2 F_V) \cdot (H_V - a_3 F_V) \equiv H_V^2 - (a_2 + a_3) H_V \cdot F_V,$$

hence

$$D \equiv (2a_1 + k)[l] + 2[s_0].$$

The argument in the previous section implies the following three inequalities:

(3.1.1) $2(a_1 + 2 - a_2 - a_3) \geqq 0,$

(3.1.2) $2a_1 + k \geqq 0,$

(3.1.3) $8(6 - \sum_{i=1}^{3} a_i - 2k) > 0,$

from (2.3.1), (2.3.2) and (2.3.3), respectively.

(3.2) It follows $0 \leqq a_1 \leqq a_2 \leqq a_3 \leqq 2$ from (3.1.1), and hence $0 < \sum_{i=1}^{3} a_i < 6$. Furthermore, we have the following seven possibilities of triples (a_1, a_2, a_3):

(3.2.1) $(a_1, a_2, a_3) = (0, 0, 0),$

(3.2.2) $(a_1, a_2, a_3) = (0, 0, 1),$

(3.2.3) $(a_1, a_2, a_3) = (0, 0, 2),$

(3.2.4) $(a_1, a_2, a_3) = (0, 1, 1),$

(3.2.5) $(a_1, a_2, a_3) = (1, 1, 1),$

(3.2.6) $(a_1, a_2, a_3) = (1, 1, 2),$ *and*

(3.2.7) $(a_1, a_2, a_3) = (2, 2, 2).$

In each case (3.2.3), (3.2.6), or (3.2.7), the equality holds in (3.1.1), hence $(-K_V \cdot D) = 0$ and D is a (-2)-curve. On the other hand, the ruled surface S, associated to the exact sequence $E \to O \oplus O(a_1) \to 0$ of sheaves, has a family of dimension$\geqq 1$ in each case. This contradicts the assumption that V has only finite (-2)-curves. Thus these cases are excluded.

Based on the description in (3.1.2) and (3.1.3) we can restrict the value of k as follows:

$k = 0, 1,$ or $2,$ in case (3.2.1);

$k = 0, 1,$ or $2,$ in case (3.2.2);

$k=0$, or 1, in case (3.2.4); and
$k=-2, -1, 0$, or 1, in case (3.2.5).

(3.3) Thus we obtain the following possibilities of numbers a_1, a_2, a_3, k, and $(-K_V^3)$:

Nos.	a_1	a_2	a_3	k	V	$-K_V$	D	$(-K_V^3)$
1	1	1	1	-2	$2H-2F$	$2H_V+F_V$	$2s_0$	56
2	0	0	0	0	$2H$	$2H_V+2F_V$	$2s_0$	48
3	0	0	1	0	$2H$	$2H_V+F_V$	$2s_0$	40
4	1	1	1	-1	$2H-F$	$2H_V$	$l+2s_0$	40
5	0	0	0	1	$2H+F$	$2H_V+F_V$	$l+2s_0$	32
6	0	1	1	0	$2H$	$2H_V$	$2s_0$	32
7	0	0	1	1	$2H+F$	$2H_V$	$l+2s_0$	24
8	1	1	1	0	$2H$	$2H_V-F_V$	$2l+2s_0$	24
9	0	0	0	2	$2H+2F$	$2H_V$	$2l+2s_0$	16
10	0	1	1	1	$2H+F$	$2H_V-F_V$	$l+2s_0$	16
11	0	0	1	2	$2H+2F$	$2H_V-F_V$	$2l+2s_0$	8
12	1	1	1	1	$2H+F$	$2H_V-2F_V$	$3l+2s_0$	8

In this table, the columns "V" and "$-K_V$" denote a linear equivalence class of V and $-K_V$, respectively, and the column "D" a numerical class of D. We omit [and] for the sake of simplicity.

In the rest of this section, we consider the realization of each above possibilities.

(3.4) We now exclude three possibilities, Nos.1, 2, and 11, in the table (3.3).

(3.4.1) *Exclusion of No.1.* This is the case that $E = O \oplus O(1)^{\oplus 3}$ and that $V \subset X = \mathbb{P}(E)$ is a member of $|2H-2F|$. But this case is to be excluded, because any member of the linear system $|2H-2F|$ has singularities along a section $s_0 \subset \mathbb{P}(E)$ associated to $E \to O \to 0$.

(3.4.2) *Exclusion of No.2.* This is the case that $E = O^{\oplus 4}$ and that $V \subset X = \mathbb{P}(E)$ is a member of $|2H|$. In this case, V is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle and its Picard number is three. Hence this is an excluded case.

(3.4.3) *Exclusion of No.11.* In this case, $E = O^{\oplus 3} \oplus O(1)$, $V \subset X = \mathbb{P}(E)$ is a member of $|2H+2F|$, and $-K_V \sim 2H_V - F_V$. Let $L = \mathbb{P}(O^{\oplus 3}) \cong \mathbb{P}^2 \times \mathbb{P}^1$ be a sub-$\mathbb{P}^2$-bundle of X, and $S = L \cap V$. If V is sufficiently general, S is a smooth surface. Then the linear system $|-K_V||_S$ defines a fiber structure on S because of

$$((-K_V)^2 \cdot S) = 4(H_V^3) - 8(H_V^2 \cdot F_V) = 0.$$

Therefore the number of (-2)-curves on $S \subset V$ is infinite, which contradicts the assumption of the finiteness of (-2)-curves.

(3.5) Next, we construct the quadric bundle with our conditions in the rest of the cases in the table (3.3). To do this, we have only to confirm that there exists a smooth member V of the linear system $|2H-kF|$, that V has only a finite number of (-2)-curves, and that the Picard number $\rho(V)$ of V is two. In addition, we show the properties of Supplement (1.5).

(3.5.1) *Construction from the data of No.3.* In this case, we have $E=O^{\oplus 3}\oplus O(1)$ and $X=\mathbb{P}(E)$. The general member V of $|2H|$ is a smooth Fano 3-fold with $-K_V \sim 2H_V + F_V$ and $\rho(V)=2$ by (2.5.5). Let $L=\mathbb{P}\left(O^{\oplus 3}\right) \cong \mathbb{P}^2 \times \mathbb{P}^1$ be a sub-$\mathbb{P}^2$-bundle of X. Then the first projection p_1: $L \to \mathbb{P}^2$ maps the intersection $L \cap V$ onto a smooth conic C. For any $c \in C$, $s=p^{-1}(c) \cong \mathbb{P}^1$ and $(s \cdot H_V)=0$, hence the class $[s]$ generates an edge of a Mori cone $\overline{NE}(V)$ of V. The edge is an extremal ray of E_1-type, and the corresponding linear system is $|H_V|$. Since $\dim|H_V|=5$ and $(H_V^3)=2$, the contraction morphism is a blowing-up of $\mathbb{Q}^3$ along a smooth conic. This is (1.4.1) and (1.5.1).

(3.5.2) *Construction from the data of No.4.* We have $E=O \oplus O(1)^{\oplus 3}$, $X=\mathbb{P}(E)$ and $V \sim 2H-F$. The general member V of $|2H-F|$ is smooth, and $-K_V \sim 2H_V$. Let s_0 be the minimal section associated to the exact sequence $E \to O \to 0$. Then V contains s_0 because $(V \cdot s_0)_X=-1$; $\mu(s_0)=0$; and the class $[s_0]$ generates an edge $R=\mathbb{R}_+[s_0] \subset \overline{NE}(V)$. Since $(R \cdot -K_V)=0$, the edge R is not an extremal ray, and the irreducible curves whose classes are in R should be flopped. Any curve in V can be considered a curve in X, and X has only one irreducible reduced curve whose class is contained in R, which is s_0, hence V has also only one (-2)-curve s_0. After flopping along s_0, we have again a weak Fano 3-fold V' with an extremal ray R'. We can see that there is no curve C' with slope $\mu(C')<1$ as follows: let $a=(H_V \cdot C)$, $b=(F_V \cdot C)$, and $n_0=\#\{C \cap s_0\}$ for any effective curve $C \subset V$; then $(H_V-F_V \cdot C)=a-b \geqq n_0$ because $Bs|H_V-F_V|=s_0$; hence the strict transform C' of C has the slope $\mu(C')=\dfrac{a}{b+n_0} \geqq 1$ from (2.2.1). The ray R' is generated by the classes of the strict transforms of intersection curves C_λ between V and $S_\lambda \subset X$. Here $S_\lambda=\mathbb{P}\left(O \oplus O(1)\right)$ are ruled surfaces associated to $E \to O \oplus O(1) \to 0$ and parametrized by $\lambda \in \mathbb{P}^2$. Thus R' is of C-type (indeed C_1-type) and V' has a conic bundle structure defined by $|H_{V'}-F_{V'}|$.

The flop $V \cdots \to V'$ can be extended to the ambient varieties; a 4-dimensional flop of $X=\mathbb{P}(E)$ along the minimal section s_0 as follows. Make a blowing-up f: $\tilde{X} \to X$ of X along s_0, then $\tilde{X}$ has an exceptional divisor $E=f^{-1}(s_0) \cong \mathbb{P}^1 \times \mathbb{P}^2$, whose normal bundle is isomorphic to $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^2}(-1,-1)$. Hence E can be contracted to another direction and the morphism f': $\tilde{X} \to X'$ is obtained. Here the image X' is the $\mathbf{P}^2$-bundle $\mathbf{P}\left(\mathcal{O}_{\mathbf{P}^2}\oplus\mathcal{O}_{\mathbf{P}^2}(1)^{\oplus 2}\right)$ over $\mathbf{P}^2$. Then V' is realized by a very ample divisor in X'. Since the relative Picard number of $X' \to \mathbf{P}^2$ is one, it follows $\rho(V') = \rho(X') = 2$ by (2.5.3). Thus we have $\rho(V) = 2$. This is (1.4.2) and (1.5.2).

(3.5.3) *Construction from the data of No.5.* In this case, $E = \mathcal{O}^{\oplus 4}$, $X = \mathbf{P}^3 \times \mathbf{P}^1$ and $V \sim 2H + F$ is a smooth Fano 3-fold with $-K_V \sim 2H_V + F_V$ of $\rho(V) = 2$ by (2.5.3). The 3-fold V is defined by an equation $Q_0(x)y_0 + Q_1(x)y_1$, where $x = [x_0 : x_1 : x_2 : x_3]$ and $y = [y_0 : y_1]$ are homogeneous coordinates of $\mathbf{P}^3$ and $\mathbf{P}^1$, respectively, and $Q_i(x)$ are quadratic forms in x. Let C be the set $\{\alpha = [\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3] \mid Q_0(\alpha) = Q_1(\alpha) = 0\}$; the set C is a quartic elliptic curve in $\mathbf{P}^3$ if V is general. For any $\alpha \in C$, $s_\alpha = \{\alpha\} \times \mathbf{P}^1 \subset V$ has a slope $\mu(s_\alpha) = 0$, hence its class generates an extremal ray R of $NE(V)$. Every curve whose class is in R is a section in V and is s_α as above. The linear system corresponding to R is $|H_V|$ with dim $|H_V| = 3$ and $(H_V^3) = 1$. Thus the contraction morphism is the restirction $p|_V : V \to \mathbf{P}^3$ of the first projection $p : X \to \mathbf{P}^3$, i.e., a blowing-up $\mathbf{P}^3$ along a quartic elliptic curve C. This is (1.4.3) and (1.5.3).

(3.5.4) *Construction from the data of No.6.* In this case, $E = \mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1)^{\oplus 2}$, $X = \mathbf{P}(E)$ and $V \sim 2H$ is a smooth weak Fano 3-fold with $-K_V \sim 2H_V$ of $\rho(V) = 2$ by (2.5.6). Let $S = \mathbf{P}\left(\mathcal{O}^{\oplus 2}\right) \cong \mathbf{P}^1 \times \mathbf{P}^1$ be a ruled surface in X corresponding to $E \to \mathcal{O}^{\oplus 2} \to 0$. The intersection $V \cap S$ is a pair of sections, i.e., $s_0 + s_1$, and these classes generate an edge $R \subset \overline{NE}(V)$ such that $\mu(R) = 0$. Any curve whose class is in R is nothing but a section in S. Thus V has only two (-2)-curves s_0, s_1 and has a $(-F_V)$-flop V', which is again a weak Fano 3-fold with a unique extremal ray $R' \subset \overline{NE}(V')$.

Now we will find curves generating the ray R'. We fix a trivial sheaf $L \cong \mathcal{O}$ on $\mathbf{P}^1$ such that an exact sequence $E \to L \to 0$ associates to the section s_0. Consider a family of ruled surfaces T_λ associated to $E \to L \oplus \mathcal{O}(1) \to 0$ parametrized by $\lambda \in \mathbf{P}^1$. Then we have a family of curves $C_\lambda = T_\lambda \cap V$ on V. Since L associates to s_0, each curve C_λ is decomposed into $s_0 + D_\lambda$. Each D_λ meets s_0 at a point from $(D_\lambda \cdot s_0)_{T_\lambda} = 1$. Considering the possibility for decomposition of D_λ, we can see that there are four points on $\mathbf{P}^1$ where $D_\lambda = D_{\lambda,0} + l_0$. Here l_0 is a line in a fiber $\mathbf{P}^3$. Since $D_\lambda \equiv 2[l] + [s_0]$ and $D_{\lambda,0} \equiv [l] + [s_0]$, the strict transforms D'_λ, $D'_{\lambda,0}$ under the $(-F_V)$-flop are equivalent to $2[l'] + 2[s_0']$, $[l'] + [s_0']$, respectively, therefore $\mu(D'_\lambda) = \mu(D'_{\lambda,1}) = 1$. Similarly as in (3.5.2), we can see that $\mu(C') \geqq 1$ for any curve $C' \subset V'$ by using (2.2.1) for $|H_V - F_V|$ with $Bs\ |H_V - F_V| = s_0 + s_1$. Thus the extremal ray R' is generated by D'_{λ} (or $D'_{\lambda,0}$). Therefore R' is of type D_2, the corresponding linear system is $|H'_V - F'_V|$, and V' has a quadric bundle structure again. This is (1.4.4) and (1.5.4).

(3.5.5) *Construction from the data of No.7.* One has $E = O^{\oplus 3} \oplus O(1)$ and $X = \mathbb{P}(E)$. The general member V of $|2H+F|$ is smooth, $-K_V \sim 2H_V$, and $\rho(V) = 2$ by (2.5.3). Let $L = \mathbb{P}(O^{\oplus 3}) \cong \mathbb{P}^2 \times \mathbb{P}^1$ be a projective subbundle in X. The intersection $V \cap L$ is defined by an equation $Q_0(x)y_0 + Q_1(x)y_1$, where x $= [x_0 : x_1 : x_2]$ and y $= [y_0 : y_1]$ are homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^1$, respectively. For any $\alpha \in A = \{\alpha = [\alpha_0 : \alpha_1 : \alpha_2] \mid Q_0(\alpha) = Q_1(\alpha) = 0\}$, $s_\alpha = \{\alpha\} \times \mathbb{P}^1$ is in V and $\mu(s_\alpha) = 0$, hence its class generates an edge $R \subset \overline{NE}(V)$. Similarly as in (3.5.3), each curve whose class is in R is the same s_α as stated above. For general V, $\# A = 4$, hence V has just four (-2)-curves $s_1,\cdots,s_4$ and has a $(-F_V)$-flop V'.

We will fix an extremal ray of $\overline{NE}(V')$ in a manner similar to (3.5.2). Using (2.2.1) for $|2H_V - F_V|$ with $Bs\ |2H_V - F_V| = \cup_{i=1}^{4} s_i$, we have $\mu(C') \geqq 1$ for any curve $C' \subset V'$. On the other hand, two sections s_1, s_2 define a ruled surface $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ in X; its intersection with V is a reducible curve $s_1 + s_2 + l_0$ with three irreducible components; and $\mu(l_0') = 2$. Hence the extremal ray R' has a slope $\mu(R') = 2$. The contraction morphism of R' is defined by a linear system $|2H_{V'} - F_{V'}|$; dim $|2H_{V'} - F_{V'}| = 5$ and $(2H_{V'} - F_{V'})^3 = 4$. Thus V' has an extremal ray of type E_1, and has a morphism to B_4 which is a blowing-up at a point of B_4, where B_4 is a Fano 3-fold of index 2, a complete intersection of two quadrics in $\mathbb{P}^5$. Moreover, an exceptional divisor E' of the morphism is a unique member of $|H_{V'} - F_{V'}|$, and this is a strict transform of $V \cap L$ under the $(-F_V)$-flop. This is (1.4.5) and (1.5.5).

(3.5.6) *Construction from the data of No.8.* In this case, we have $E = O \oplus O(1)^{\oplus 3}$, $X = \mathbb{P}(E)$, and V is a member of $|2H|$ with $-K_V \sim 2H_V - F_V$ and $\rho(V) = 2$ by (2.5.7). The minimal section s_0 corresponding to $E \to O \to 0$ is not contained in V, because if it was in V then $-K_V$ would not be nef by $(-K_V \cdot s_0) = -1$. The general member $V \in |2H|$ is smooth and does not contain s_0. Now consider a family of ruled surfaces $S_\lambda = \mathbb{P}(O \oplus O(1)) \subset X$ associated to $E \to O \oplus O(1) \to 0$, and parametrized by $\lambda \in \mathbb{P}^2$. Each S_α contains the minimal section s_0, and we have Pic$(S_\lambda) \cong \mathbb{Z}[s_0] \oplus \mathbb{Z}[l] \cong \mathbb{Z}[H_\lambda] \oplus \mathbb{Z}[F_\lambda]$ for restrictions H_λ, F_λ of H, F to S_λ. Here $l \sim F_\lambda$ and $H_\lambda \sim s_0 + l$ on S_λ. If the intersection curve $D_\lambda = V \cap S_\lambda \equiv 2[s_0] + 2[l]$ is decomposed into $D_{\lambda,1} + D_{\lambda,2}$, then there are the following two possibilities:

(3.5.6.1) $\quad D_{\lambda,1} \equiv 2[s_0] + [l]$ and $D_{\lambda,2} \equiv [l]$, or

(3.5.6.2) $\quad D_{\lambda,1} \equiv D_{\lambda,2} \equiv [s_0] + [l]$

since s_0 cannot be a component of D_λ. The case (3.5.6.1) cannot occur because $D_{\lambda,1}$ has s_0 as an irreducible component by $(D_{\lambda,1}\cdot s_0)_{S_\lambda} = -1$. The case (3.5.6.2) occurs for λ on a plane curve C_4 of degree 4; each $D_{\lambda,i}$ is no longer decomposable. The slope of an irreducible D_α or of $D_{\lambda,i}$ in (3.5.6.2) is $\mu(D_\lambda) = \mu(D_{\lambda,i}) = 1$. Any curve of slope $\leqq 1$ is in a ruled surface S_λ by (2.4), thus $\overline{NE}(V)$ has an edge R of slope 1 generated by $[D_{\lambda,i}]$ (or $[D_\lambda]$). Since $-K_V \sim 2H_V - F_V$, R is an extremal ray and V is a Fano 3-fold. The contraction morphism associated to R is defined by $|H_V - F_V|$ with dim $|H_V - F_V| = 3$. Hence V has a conic bundle structure over $\mathbb{P}^2$ with a discriminant locus C_4 of degree 4. This is (1.4.6) and (1.5.6).

(3.5.7) *Construction from the data of No.9.* One has $E = O^{\oplus 4}$ and $X = \mathbb{P}^3 \times \mathbb{P}^1$. Each member V of $|2H+2F|$ is defined by an equation $Q_0(x)y_0^2 + Q_1(x)y_0y_1 + Q_2(x)y_1^2$ as in (3.5.3); $\rho(V) = 2$ by (2.5.3). Consider the set $A = \{\alpha = [\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3] \mid Q_0(\alpha) = Q_1(\alpha) = Q_2(\alpha) = 0\}$ in $\mathbb{P}^3$, which consists of eight points for general V. For each $\alpha \in A$, the corresponding section $s_\alpha = \{\alpha\} \times \mathbb{P}^1$ is in V and $\mu(s_\alpha) = 0$, hence its class generates an edge of $\overline{NE}(V)$. Similarly as in (3.5.3) one can see that the (-2)-curves on V are the same s_α as stated above. Hence, for the general member $V \in |2H+2F|$, V is a smooth weak Fano 3-fold with just eight (-2)-curves and has a $(-F_V)$-flop V'. The variety V' is again a weak Fano 3-fold with only one extremal ray R' of $\overline{NE}(V')$. To find $\mu(R')$, we use (2.2.1) as in (3.5.5) for $|2H_V - F_V|$. Then we can see that the type of R' is D_2 and that the contraction morphism is defined by $|2H_{V'} - F_{V'}|$. This is (1.4.7) and (1.5.7).

(3.5.8) *Construction from the data of No.10.* We have $E = O^{\oplus 2} \oplus O(1)^{\oplus 2}$ and $X = \mathbb{P}(E)$. The general member V of $|2H+F|$ is a smooth variety with $-K_V \sim 2H_V - F_V$, and does not contain minimal sections s associated to $E \to O \to 0$; $\rho(V) = 2$ by (2.5.3). Let $S = \mathbb{P}(O^{\oplus 2}) \cong \mathbb{P}^1 \times \mathbb{P}^1$ be a ruled surface in X and consider the intersection curve $C = V \cap S$; C is an irreducible curve with $(-K_V\cdot C) = 0$ and $\mu(C) = 1/2$. Using a similar argument as in (2.4) shows that this curve is the only (-2)-curve in V. Hence V is a weak Fano 3-fold with only one (-2)-curve and has a $(-F_V)$-flop V'; the 3-fold V' is again a weak Fano 3-fold, and has a unique extremal ray R' of $\overline{\mathrm{NE}}(\mathrm{V}')$. By (2.2.1), $\mu(R') = 1$, hence the contraction morphism is defined by $|H_{V'} - F_{V'}|$ with dim $|H_{V'} - F_{V'}| = 1$ and $(2H_{V'} - F_{V'})\cdot(-K_{V'})^2 = 4$, and R' is of type D_1. Therefore V' has a del Pezzo fibration of degree 4. This is (1.4.8) and (1.5.8).

(3.5.9) *Construction from the data of No.12.* In this case, we have $E = O \oplus O(1)^{\oplus 3}$ and $X = \mathbb{P}(E)$. The general member V of $|2H+F|$ is a smooth variety with $-K_V \sim$

$2H_V - 2F_V$, $\rho(V) = 2$, and does not contain the minimal section s_0. We consider a family of ruled surfaces $S_\lambda = \mathbb{P}\ (O \oplus O(1)) \subset X$ parametrized by $\lambda \in \mathbb{P}^2$. Similarly as (3.5.6), $D_\lambda = V \cap S_\lambda \equiv 2[s_0] + 3[l]$ may be decomposed into

(3.5.9.1) $D_{\lambda,1} \equiv [s_0] + [l]$ and $D_{\lambda,2} \equiv [s_0] + 2[l]$, or

(3.5.9.2) $D_{\lambda,1} \equiv 2[s_0] + 2[l]$ and $D_{\lambda,2} \equiv [l]$.

Direct calculation using local coordinates shows that the case (3.5.9.1) occurs at just sixteen λ's, and that the case (3.5.9.2) occurs at just two λ's. We have $(-K_V \cdot D_{\lambda,1}) = 0$ in each case. The second case is easy but the first case is slightly complicated. In order to find (-2)-curves in V, we have only to consider the curves stated above for the sake of (2.4). Hence V has just sixteen (-2)-curves as sections and two (-2)-curves as bisections, and has a $(-F_V)$-flop V' with a unique extremal ray. Similarly as in the other cases, we see that the type of the ray is D_2 and that V' has a quadric bundle structure defined by $|4H_{V'} - 5F_{V'}|$. This is (1.4.9) and (1.5.9).

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