4次元代数多様体上のサイクルの交点数

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Intersection numbers of cycles on a 4-fold

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要　　旨

代数多様体上のサイクルの交点数は，数値的方法で代数多様体の性質を調べるときに重要な役割を演ずる。中でも，二次変換により生じた例外サイクルと元の多様体上のサイクルの引き戻しとの交点数は，双有理幾何学上，特に重要である。代数多様体の次元が3次元以下の場合はよく知られているので，ここでは4次元の場合の具体的な公式を導く。

Summary

Intersection numbers of cycles on a variety play a very important role in numerical methods studying varieties. In particular, after a monoidal transformation, intersection numbers between the exceptional and pulled-back cycles are essencial in birational geometry. Two- and three-dimensional cases are well-known, and this note deals with the four-dimensional case and derives explicit formulae for that case.

Received Apr. 30, 1994

Key words : 4-fold, monoidal transformation, intersection number

Section 0. Introduction

Intersection numbers between cycles in a variety play a very important role in numerical approach to any problem in algebraic geometry. In particular, we are interested in the intersection numbers with exceptional cycles after a monoidal transformation. In a surface case, as is well-known, the intersection number with an exceptional curve is very simple. In a 3-fold case the table of intersection numbers with exceptional cycles can be found in some papers, for example in [Iskovskih]. In a 4-fold case the author has not seen this kind of table in any paper, so we will complete table.

To calculate the intersection numbers, we have a framework, as established in [Manin] ; thus, we can recall it as shown in Section 1. This calculation should be separated into three cases according to the dimension of the center of a monoidal transformation : the point case is treated in Section 2 ; the curve case in Section 3 ; and the surface case in Section 4. We summarize combinatorical calculation at the end of this paper, in the Appendix. The intersection tables are finally shown in (2.6), (3.5), and (4.5), respectively.

The auther would like to express his thanks to Mr. Takashi Maeda for his useful conversation and my gratitude to Prof. Shigeru Mukai for his encouragement.

Section 1. Preliminary and fundamental results

(1.0) In this section we recall fundamental results on intersection numbers following [Manin].

(1.1) Let X be a regular noetherian scheme with an ample invertible sheaf, where a scheme X is called regular if there exists an integer n such that each coherent sheaf on X has locally everywhere a free resolution of length$\leqq$n, equivalently if all of its local rings are regular local rings.

We now introduce the Grothendieck group K(X) of X. Let S be the category of coherent sheaves on X and L its full subcategory of locally free sheaves. Let $Z[S]$ (resp. $Z[L]$) denote the free abelian group generated by the classes $[\mathcal{F}]$ up to isomorphism of sheaves $\mathcal{F}$ in S (resp. in L). For each exact seqence

$$0\to\mathcal{F}_1\to\mathcal{F}_2\to\mathcal{F}_3\to 0$$

of sheaves in S (resp. in L), we form the element

$$[\mathcal{F}_2]-[\mathcal{F}_1]-[\mathcal{F}_3]$$

in $Z[S]$ (resp. in $Z[L]$) and consider the subgroup I (resp. J) generated by these elements in $Z[S]$ (resp. in $Z[L]$). If X is regular with an ample invertible sheaf, then

$$Z[S]/I=Z[L]/J,$$

and this quotient group is called the Grothendieck group K(X) of X ([Manin, 1.9]). We denote by $\mathrm{cl}(\mathcal{F})$ the class corresponding to $\mathcal{F}$ in K(X).

(1.2) The group K(X) becomes a commutative ring with identity $1=\mathrm{cl}(\mathcal{O}_X)$ when multiplication is defined by

$$\mathrm{cl}(\mathcal{F}_1)\cdot\mathrm{cl}(\mathcal{F}_2)=\mathrm{cl}(\mathcal{F}_1\otimes\mathcal{F}_2).$$

This multiplication corresponds to the intersection of cycles as follows :

$$\mathrm{cl}(\mathcal{O}_{Y1})\cdot\mathrm{cl}(\mathcal{O}_{Y2})=\mathrm{cl}(\mathcal{O}_{Y1\cap Y2}).$$

(1.3) A closed subscheme Y in X is said to be regularly embedded if for each point y of Y there exists an affine neighbourhood U=Spec A such that the ideal I of A defining Y in U is generated by an A-regular sequence.

Let Y be a regularly embedded closed subscheme in X with defining ideal I and $\mathcal{O}_Y$

$=\mathcal{O}_X/I$ its structure sheaf. Then the conormal sheaf

$\mathcal{C}=I/I^2|_Y$

is locally free $\mathcal{O}_Y$-module, and its rank at y on Y is equal to the local number of equations defining Y in the neighbourhood of this point.

(1.4) Theorem ([Manin, 3.10]). For any regular noetherian scheme X there exist uniquely defined mappings

$\lambda^i : K(X) \to K(X) \quad (i=0, 1, 2, \ldots)$,

satisfying the conditions

$\lambda^i(cl(F)) = cl(\Lambda^i(F))$

for any locally free $\mathcal{O}_X$-module F.

(1.4.1) Particularly, $\lambda^0(x)=1$ and $\lambda^1(x)=x$ for any $x \in K(X)$. For any $x \in K(X)$ we now put

$\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x) \cdot t^i$;

then we have $\lambda_t(x+y)=\lambda_t(x) \cdot \lambda_t(y)$ for any two elements x, $y \in K(X)$.

(1.5) Proposition (Projection Formula, [Manin, 7.13]). Let $f : X \to Y$ be a projective morphism of regular noetherian schemes, $x \in K(X)$, and $y \in K(Y)$. Then

$f_*(x \cdot f^*(y)) = f_*(x) \cdot y$.

(1.6) We next introduce a filtration into the ring K(X) as follows.

(1.6.1) The operations

$\gamma^i : K(X) \to K(X) \quad (i=0, 1, 2, \ldots)$

are defined by

$\gamma_t(x) = \sum_{i=0}^{\infty} \gamma^i(x) \cdot t^i = \lambda_{t/(1-t)}(x) = \sum_{i=0}^{\infty} \lambda^i(x)(t+t^2+\cdots)^i$

in the multiplicative group $1+t \cdot K[[t]]$.

(1.6.2) The following equarities immediately from the definition :

(a) $\gamma^0(x)=\lambda^0(x)=1$, $\gamma^1(x)=\lambda^1(x)=x$ for any $x \in K(X)$;

(b) $\gamma_t(x+y)=\gamma_t(x) \cdot \gamma_t(y)$ for any two x, $y \in K(X)$;

(c) $\gamma_t(\ell-1)=1+(\ell-1) \cdot t$ for the class ℓ of an invertible sheaf, i.e., $\gamma^0(\ell-1)=1$, $\gamma^1(\ell-1)=\ell-1$, and $\gamma^i(\ell-1)=0$ for any integer $i \geqq 2$.

(1.6.3) We define a filtration of the ring K(X) as follows : the Z-submodule $F^1K(X)$ is generated by elements $cl(F)$-r for each locally free sheaf F of rank r, and the Z-submodules

$F^nK(X)$ is generated by elements $\gamma_1(x_1)^{r_1}\cdots\gamma_k(x_k)^{r_k}$, where $x_i\in F^1K(X)$ and $\sum_{i=1}^{k} r_i\geqq n$. Now set $G^iK(X)=F^iK(X)/F^{i+1}K(X)$ and $GK(X)=\sum_{i=0}^{\infty}G^iK(X)$.

(1.6.4) It is obvious that $F^iK(X)\cdot F^jK(X)\subset F^{i+j}K(X)$.

(1.6.5) Theorem ([Manin, 9.1]). Let X be a regular noetherian scheme of dimension d with an ample sheaf. Then

$F^{d+1}K(X)=0.$

(1.6.6) For any locally free sheaf F of rank r on X, we get an expression of γ :

$$\gamma^i(\mathrm{cl}(F)-r)=\sum_{k=0}^{i}(-1)^k\begin{bmatrix} r-i+k \\ k\end{bmatrix}\lambda^{i-k}(\mathrm{cl}(F)),$$

where $[\ \ ]$ is the binomial coefficient (see Appendix). A proof of this equality will be given in Appendix (A.6).

(1.6.7) For any locally free sheaf F of rank r on X

$\lambda_{-1}(\mathrm{cl}(F))=(-1)^r\gamma^r(\mathrm{cl}(F)-r).$

Indeed, by (1.6.6) we have

$$\gamma^r(\mathrm{cl}(F)-r)=\sum_{i=0}^{r}(-1)^i\begin{bmatrix} r-r+i \\ i\end{bmatrix}\lambda^{r-i}(\mathrm{cl}(F))=\sum_{i=0}^{r}(-1)^i\lambda^{r-i}(\mathrm{cl}(F)).$$

(1.7) Let F be a locally free sheaf of rank $\mathrm{rk}F$ on the scheme X. For any positive integer i, put

$c_i(F)\equiv\gamma^i(\mathrm{cl}(F)-\mathrm{rk}F) \mod F^{i+1}K(X)\in G^iK(X),$

which is called the i-th Chern class of F.

(1.7.1) For $i>\mathrm{rk}F$, $c_i(F)=0$ ([Manin, 11.2.5]). This follows from (1.6.6).

(1.7.2) For any exact sequence

$0\to E\to F\to G\to 0$

of locally free sheaves on X, we have

$c_i(F)=\sum_{k=0}^{i}c_k(E)\cdot c_{i-k}(G)$

because

$$\begin{aligned}\gamma^i(\mathrm{cl}(F)-\mathrm{rk}F)&=\gamma^i((\mathrm{cl}(E)-\mathrm{rk}E)+(\mathrm{cl}(G)-\mathrm{rk}G))\\&=\sum_{k=0}^{i}\gamma^k(\mathrm{cl}(E)-\mathrm{rk}E)\cdot\gamma^{i-k}(\mathrm{cl}(G)-\mathrm{rk}G).\end{aligned}$$

(1.7.3) Lemma. For any locally free sheaf G of rank r on X, the first Chern class of G equals to one of its highest exterior product $\Lambda^i(F)$ modulo $F^2K(X)$.

PROOF. Applying the splitting principle to F, we can assume that $\mathrm{cl}(F)=\sum_{i=1}^{r}\ell_i$. For

$x=\ell_1+\ell_2$, the equality $\lambda^2 x=\ell_1\cdot\ell_2$ deduces that

$$c_1(\lambda^2 x)=\lambda^2 x-1=(\ell_1-1)\cdot(\ell_2-1)+\ell_1+\ell_2-2\equiv c_1(x).$$

Using the above fact recursively, we prove this lemma in general case.

(1.8) For any morphism $f: X\to Y$ of regular noetherian schemes, the pull-back f^* operates compatibly with the above filtration. Indeed, we have that

$$f^*(\Lambda^i(\mathcal{F}))=\Lambda^i(f^*(\mathcal{F}))$$

for any locally free sheaf $\mathcal{F}$ on Y, and hence that

$$f^*(\lambda_t(y))=\lambda_t(f^*(y))$$

for any element $y\in K(Y)$. Thus the definition of γ follows that

$$f^*(\gamma_t(y))=f^*(\lambda_{t/(1-t)}(y))=\lambda_{t/(1-t)}(f^*(y))=\gamma_t(f^*(y)),\quad \text{i.e.,}\quad f^*(\gamma^i(y))=\gamma^i(f^*(y))$$

for each coefficient of t^i. Therefore we have natural homomorphisms

$$f^*: F^iK(Y)\to F^iK(X)$$

of $\mathbb{Z}$-modules, and an induced ring homomorphism

$$Gf^*: GK(Y)\to GK(X).$$

Moreover we get

$$f^*(c_i(\mathcal{F}))=c_i(f^*(\mathcal{F}))$$

for the i-th Chern classes of locally free sheaf $\mathcal{F}$ on Y.

(1.9) Let Y be a regularly embedded closed subscheme of a regular noetherian scheme X, $f: X'\to X$ a monoidal transformation with center Y, and Y' an exceptional divisor:

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f.} \\ Y & \xrightarrow{i} & X \end{array}$$

Let $\mathcal{C}$ and $\mathcal{C}'$ be the conormal sheaves of Y and Y' respectively. Then $g: Y'\to Y$ is a projective bundle of relative dimension $\mathrm{rk}\,\mathcal{C}-1$.

(1.9.1) Proposition ([Manin, 13.4, 13.5]). For any element $y\in K(Y)$ we have

$$i^*i_*(y)=y\cdot\lambda_{-1}(\mathrm{cl}(\mathcal{C})).$$

Moreover for any two elements $y_1, y_2\in K(Y)$

$$i_*(y_1)\cdot i_*(y_2)=i_*(y_1\cdot y_2\cdot\lambda_{-1}(\mathrm{cl}(\mathcal{C})))\quad \text{in } K(X).$$

(1.9.2) Proposition ([Manin, 13.7]). Let $\mathcal{F}$ be the kernel of a canonical epimorphism $g^*\mathcal{C}\to\mathcal{C}'$. Then for each element $y\in K(Y)$ we have

$$f^*i_*(y)=j_*(g^*(y)\cdot\lambda_{-1}(\mathrm{cl}(\mathcal{F})))\quad \text{in } K(X').$$

(1.10) Let Y be a regularly embedded closed subscheme of a regular noetherian scheme

X, i : Y→X, $\mathcal{C}$ a conormal sheaf of Y, and Ω_X (resp. Ω_Y) a sheaf of differentials of X (resp. Y). There is a natural exact sequence

$$0 \to \mathcal{C} \to i^*(\Omega_X) \to \Omega_Y \to 0$$

of sheaves on Y, hence it follows from (1.7.2) and (1.7.3) that

$$c_1(\mathcal{C}) = i^*(c_1(\Omega_X)) - c_1(\Omega_Y) \equiv i^*(c_1(\Lambda^n\Omega_X)) - c_1(\Lambda^m\Omega_Y) = i^*(c_1(-K_X)) - c_1(-K_Y) \bmod F^2K(X).$$

Section 2. The case where the center is a point

(2.0) From this section to the end of this paper, we consider the monoidal transformations of only four-dimensional varieties and fix the following notation ; let Y be a regular subvariety of a regular 4-fold X, f : X'→X a monoidal transformation with center Y, and Y' an exceptional divisor for f ;

$$\begin{array}{ccc} Y' & \xrightarrow{j} & X' \\ g\downarrow & & \downarrow f. \\ Y & \xrightarrow{i} & X \end{array}$$

Let $\mathcal{C}$ and $\mathcal{C}'$ be the conormal sheaves of Y and Y' respectively, and $\ell = \mathrm{cl}(\mathcal{C}')$ in K(Y'). We want to calculate the intersection numbers between Y' and pulled-back cycles. For a while we consider all cases, without the dimensional restriction on the center of monoidal transformation.

(2.1) Considering an exact sequence

$$0 \to \mathcal{F} \to g^*\mathcal{C} \to \mathcal{C}' \to 0,$$

we have

$$g^*(c_i(\mathcal{C})) \equiv c_i(g^*(\mathcal{C})) \equiv \sum_{k=0}^{i} c_k(\mathcal{F}) \cdot c_{i-k}(\mathcal{C}') \equiv c_i(\mathcal{F}) + c_{i-1}(\mathcal{F}) \cdot c_1(\ell)$$

by (1.8), (1.7.2), and (1.7.1). Consequently

(2.1.1) $\quad -c_1(\ell) \equiv -g^*(c_1(\mathcal{C})) + c_1(\mathcal{F}),$

(2.1.2) $\quad -c_1(\ell) \cdot c_1(\mathcal{F}) \equiv -g^*(c_2(\mathcal{C})) + c_2(\mathcal{F}),$ and

(2.1.3) $\quad -c_1(\ell) \cdot c_2(\mathcal{F}) \equiv -g^*(c_3(\mathcal{C})) + c_3(\mathcal{F}).$

(2.2) Applying Proposition (1.9.1) to a regular embedding j : Y'→X', and replacing y with the class $1_{Y'} = \mathrm{cl}(\mathcal{O}_{Y'})$ corresponding to a cycle Y' in K(Y'), we have

$$j_*(1_{Y'}) \cdot j_*(1_{Y'}) = j_*(1_{Y'} \cdot 1_{Y'} \cdot \lambda_{-1}(\ell)) = j_*(\lambda_{-1}(\ell))$$

in K(X'). Since the class $\ell = \mathrm{cl}(\mathcal{C}')$ corresponds to an invertible sheaf, we get

$$\lambda_{-1}(\ell) \equiv -c_1(\ell)$$

by (1.6.7) and hence

(2.2.1) $(j_*(1_{Y'}))^2 \equiv -j_*(g^*c_1(C)) + j_*(c_1(F))$

by (2.1.1). Therefore it follows from (2.1.1)-(2.1.3) that

$$\begin{aligned}
(2.2.2)\quad (j_*(1_{Y'}))^3 &\equiv -j_*(g^*c_1(C)\cdot(-c_1(\ell))) + j_*(c_1(F)\cdot(-c_1(\ell))) \\
&\equiv j_*(g^*(c_1(C))^2) - j_*(g^*(c_1(C))\cdot c_1(F)) - j_*(g^*(c_2(C))) + j_*(c_2(F)) \\
&\equiv j_*(g^*(c_1(C)^2 - c_2(C))) - j_*(g^*(c_1(C))\cdot c_1(F)) + j_*(c_2(F)) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
(2.2.3)\quad (j_*(1_{Y'}))^4 &\equiv j_*(-g^*c_1(C)^2\cdot c_1(\ell)) - j_*(-g^*c_2(C)\cdot c_1(\ell)) \\
&\quad - j_*(-g^*c_1(C)\cdot c_1(F)\cdot c_1(\ell)) + j_*(-c_2(F)\cdot c_1(\ell)) \\
&\equiv -j_*(g^*c_1(C)^3) + j_*(g^*c_1(C)^2\cdot c_1(F)) + j_*(g^*(c_2(C)\cdot c_1(C))) \\
&\quad - j_*(g^*c_2(C)\cdot c_1(F)) + j_*(g^*(c_1(C)\cdot c_2(C))) \\
&\quad - j_*(g^*c_1(C)\cdot c_2(F)) - j_*(g^*(c_3(C))) + j_*(c_3(F)) \\
&\equiv -j_*(g^*(c_1(C)^3 + 2c_1(C)\cdot c_2(C) - c_3(C))) + j_*(g^*(c_1(C)^2 - c_2(C))\cdot c_1(F)) \\
&\quad - j_*(g^*(c_1(C))\cdot c_2(F)) + j_*(c_3(F)).
\end{aligned}$$

(2.3) In the rest of this section we treat the case where the center Y of monoidal transformation is a point. This is the simplest case.

(2.3.1) Since the center Y is a point, $F^1K(Y)=0$ by Theorem (1.6.5) and hence $c_1(C) \equiv c_2(C) \equiv c_3(C) \equiv 0$ in GK(Y).

(2.4) Thus relations (2.1.1)-(2.1.3) are following:

(2.4.1) $c_1(F) \equiv -c_1(\ell)$,

(2.4.2) $c_2(F) \equiv -c_1(\ell)\cdot c_1(F) \equiv c_1(\ell)^2$, and

(2.4.3) $c_3(F) \equiv -c_1(\ell)\cdot c_2(F) \equiv -c_1(\ell)^3$;

and (2.2.1)-(2.2.3) become as below:

(2.4.4) $(j_*(1_{Y'}))^2 \equiv j_*(c_1(F)) \equiv -j_*(c_1(\ell))$,

(2.4.5) $(j_*(1_{Y'}))^3 \equiv j_*(c_2(F)) \equiv j_*(c_1(\ell)^2)$, and

(2.4.6) $(j_*(1_{Y'}))^4 \equiv j_*(c_3(F)) \equiv -j_*(c_1(\ell)^3)$.

(2.5) By (1.6.7) one has

$$c_3(F) \equiv -\lambda_{-1}(\mathrm{cl}(F))$$

and by (1.9.2)

$$(j_*(1_{Y'}))^4 \equiv -f^*(i_*(1_Y)) \equiv -1.$$

For any $x \in F^iK(X)$ $(i>0)$, since $i^*(x) \in F^iK(Y)=0$ by (1.8) and (2.3.1),

$$j_*(y')\cdot f^*(x) = j_*(y'\cdot j^*f^*(x)) = j_*(y'\cdot g^*i^*(x)) = 0,$$

where y' is any element of K(Y').

(2.6) In this case, Y' is the three dimensional projective space, then $c_1(\ell)$ is realized by a general plane $H' \subset Y'$ and $(c_1(\ell))^2$ by a line $L' \subset Y'$. Thus we consequently obtain the following :

(2.6.1) for any i-cycle Z on X

$Y' \cdot f^*(Z) = H' \cdot f^*(Z) = L' \cdot f^*(Z) = 0$;

(2.6.2) $Y'^2 = -H'$, $Y'^3 = -Y' \cdot H' = L'$ and $Y'^4 = Y' \cdot L' = H'^2 = -1$.

Section 3. The case where the center is a curve

(3.0) This section treats the case that the center Y is a curve. Since dim Y=1,

(3.0.1) $F^2K(Y) = 0$

by Theorem (1.6.4) and hence

(3.0.2) $c_2(\mathcal{C}) \equiv c_3(\mathcal{C}) \equiv (c_1(\mathcal{C}))^2 \equiv 0$ in GK(Y).

On the other, because $\mathrm{rk}\,\mathcal{C} = \mathrm{codim}_X\, Y = 3$ and $\mathrm{rk}\,\mathcal{C}' = 1$, we have $\mathrm{rk}\,\mathcal{F} = 2$ thus

(3.0.3) $c_3(\mathcal{F}) \equiv 0$ in GK(Y') by (1.7.1) and

(3.0.4) $c_2(\mathcal{F}) \equiv \lambda_{-1}(\mathrm{cl}(\mathcal{F}))$ in GK(Y') by (1.6.7).

(3.1) Thus relations (2.1.1)-(2.1.3) are following :

(3.1.1) $-c_1(\ell) \equiv -g^*(c_1(\mathcal{C})) + c_1(\mathcal{F})$,

(3.1.2) $-c_1(\ell) \cdot c_1(\mathcal{F}) \equiv c_2(\mathcal{F})$, and

(3.1.3) $-c_1(\ell) \cdot c_2(\mathcal{F}) \equiv 0$.

(3.2) From (2.2.1)-(2.2.3) it follows that

(3.2.1) $(j_*(1_{Y'}))^2 \equiv -j_*(g^*c_1(\mathcal{C})) + j_*(c_1(\mathcal{F}))$,

(3.2.2) $(j_*(1_{Y'}))^3 \equiv -j_*(g^*c_1(\mathcal{C}) \cdot c_1(\mathcal{F})) + j_*(c_2(\mathcal{F})) \equiv -j_*(g^*c_1(\mathcal{C}) \cdot c_1(\mathcal{F})) + f^*i_*(1_Y)$, and

(3.2.3) $(j_*(1_{Y'}))^4 \equiv -j_*(g^*c_1(\mathcal{C}) \cdot c_2(\mathcal{F})) \equiv -f^*i_*(c_1(\mathcal{C}))$

by (3.0.2)-(3.0.4).

(3.3) For any $x \in F^iK(X)$ $(i>1)$, we have $i^*(x) \in F^iK(Y) = 0$ by (1.8) and (3.0.1), therefore for any $y' \in K(Y')$

$j_*(y') \cdot f^*(x) = j_*(y' \cdot j^*f^*(x)) = j_*(y' \cdot g^*i^*(x)) = 0.$

For any $x \in F^1K(X)$,

$j_*(1_{Y'}) \cdot f^*(x) = j_*(1_{Y'} \cdot g^*i^*(x))$,

$j_*(c_1(\mathcal{F})) \cdot f^*(x) = j_*(c_1(\mathcal{F}) \cdot g^*i^*(x))$, and

$j_*(g^*(y)) \cdot f^*(x) = j_*(g^*(y \cdot i^*(x))) = 0.$

Here y is an element of K(Y) and $y \cdot i^*(x) \in F^2K(Y) = 0$.

(3.4) Put $p=g^*(1_P) \in K(Y')$ for a point P on Y. It is obvious that p is the class of a fiber plane H' of the projective plane bundle $g: Y' \to Y$. From (3.1.1)-(3.1.3) we have

$j_*(p) \cdot j_*(1_{Y'})=j_*(p \cdot (-c_1(\ell)))=j_*(p \cdot c_1(F))$,

$j_*(p) \cdot j_*(c_1(F))=j_*(p \cdot c_2(F))=j_*(g^*(1_P) \cdot c_2(F))=f^*i_*(1_P)$,

$j_*(c_1(F)) \cdot j_*(1_{Y'})=j_*(c_2(F))=f^*i_*(1_Y)$,

$j_*(c_1(F))^2=j_*(c_1(F) \cdot c_2(F))=j_*((-c_1(\ell)+g^*c_1(C)) \cdot c_2(F))=j_*(g^*c_1(C) \cdot c_2(F))$
$=f^*i_*(c_1(C))$, and

$j_*(p \cdot c_1(F)) \cdot j_*(1_{Y'})=j_*(p \cdot c_2(F))=f^*i_*(1_P)$.

(3.5) The class $-c_1(F)$ corresponds to some surface M' in the exceptional divisor Y', and $-p \cdot c_1(F)$ to a line $L'=M' \cap H'$ in a plane H'. Therefore we obtain the following:

(3.5.1) for any i-cycle Z on X

$Y' \cdot f^*(Z)=(Z \cdot Y)H'$,

$M' \cdot f^*(Z)=(Z \cdot Y)L'$,

$H' \cdot f^*(Z)=L' \cdot f^*(Z)=0$;

(3.5.2) $Y'^2=-\deg c_1(C)H'-M'$,

$Y'^3=-\deg c_1(C)L'+f^*(Y)$,

$Y'^4=-\deg c_1(C)$,

$Y' \cdot H'=-L'$, $Y' \cdot M'=-f^*(Y)$,

$H' \cdot M'=Y' \cdot L'=-1$, $H'^2=0$, and

$M'^2=\deg c_1(C)$,

where $\deg c_1(C)=-(K_X \cdot Y)+2-2g(Y)$ by (1.10), and g(Y) is the genus of curve Y.

Section 4. The case where the center is a surface

(4.0) This section treats the case that the center Y is a surface. In this case, dim Y= 2, $\mathrm{rk}F=1$, and

(4.0.1) $F^3K(Y)=0$

by Theorem (1.6.4), hence

(4.0.2) $c_3(C) \equiv c_2(C) \cdot c_1(C) \equiv (c_1(C))^3 \equiv 0$ in GK(Y);

(4.0.3) $c_2(F) \equiv 0$, $-c_1(F) \equiv \lambda_{-1}(\mathrm{cl}(F))$ in GK(Y') by (1.7.1) and (1.6.7).

(4.1) Relations (2.1.1)-(2.1.3) are rewritten as follows:

(4.1.1) $-c_1(\ell) \equiv -g^*(c_1(C))+c_1(F)$ and

(4.1.2) $-c_1(\ell) \cdot c_1(F) \equiv -g^*(c_2(C))$.

(From (2.1.3) we get only a trivial relation.)

(4.2) From (2.2.1)-(2.2.3) it follows that

(4.2.1) $(j_*(1_{Y'}))^2 \equiv -j_*(g^*c_1(\mathcal{C})) - f^*(i_*(1_Y))$,

(4.2.2) $(j_*(1_{Y'}))^3 \equiv j_*(g^*(c_1(\mathcal{C})^2 - c_2(\mathcal{C}))) + f^*(i_*c_1(\mathcal{C}))$, and

(4.2.3) $(j_*(1_{Y'}))^4 \equiv -f^*(i_*(c_1(\mathcal{C})^2 - c_2(\mathcal{C})))$.

(4.3) For any element $y \in K(Y)$, we obtain that

(4.3.1) $j_*(1_{Y'}) \cdot j_*g^*(y) \equiv j_*(g^*(y) \cdot (-c_1(\ell))) \equiv -j_*(g^*(y \cdot c_1(\mathcal{C}))) - f^*i_*(1_y)$

and that

(4.3.2) $j_*(1_{Y'}) \cdot f^*i_*(y) \equiv j_*(g^*(y) \cdot \lambda_{-1}(\mathcal{F}) \cdot \lambda_{-1}(\ell)) \equiv j_*(g^*(y \cdot c_2(\mathcal{C})))$ in $GK(X')$.

For any $x \in K(X)$,

(4.3.3) $j_*(1_{Y'}) \cdot f^*(x) \equiv j_*(j^*f^*(x)) \equiv j_*(g^*i^*(x))$ in $GK(X')$.

Furthermore

(4.3.4) $(i_*(1_Y))^2 \equiv i_*(\lambda_{-1}(\mathcal{C})) \equiv i_*(c_2(\mathcal{C}))$ in $GK(X)$.

(4.4) Let L' be a general fiber of the projective line bundle $g: Y' \to Y$. Then for any cycle W on Y we may write

$g^*(W) = W \times L'$,

particularly, if W is a 0-cycle (i.e., it is interpreted as a number w) then

$g^*(W) = wL'$.

The class of L' is equal to $j_*g^*(p)$ where p is a point on Y.

(4.5) To conclude, we obtain the following:

(4.5.1) for any i-cycle Z on X

$Y' \cdot f^*(Z) = (Z \cdot Y) \times L'$, and

$L' \cdot f^*(Z) = 0$;

(4.5.2) $Y'^2 = -c_1(\mathcal{C}) \times L' - f^*(Y)$,

$Y'^3 = (c_1(\mathcal{C})^2 - c_2(\mathcal{C}))L' - f^*(c_1(\mathcal{C}))$,

$Y'^4 = -(c_1(\mathcal{C})^2 - c_2(\mathcal{C}))$,

$Y' \cdot (W \times L') = -(W \cdot c_1(\mathcal{C}))_Y L' - f^*(W)$, and

$Y' \cdot L' = -1$

where $c_1(\mathcal{C}) = -K_X|_Y + K_Y$ and $c_2(\mathcal{C}) = (Y \cdot Y)$ by (1.10) and (4.3.4).

Appendix

(A.0) In this section we summarize results for combinatorial calculas. The symbol $\begin{bmatrix} a \\ b \end{bmatrix}$ is used the following sense :

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{cases} \dfrac{a!}{(a-b)!b!} & \text{if } 0 \leqq b \leqq a, \\ 0 & \text{otherwise.} \end{cases}$$

(A.1) It follows immediately that

$$\begin{bmatrix} a-1 \\ b \end{bmatrix} + \begin{bmatrix} a-1 \\ b-1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

for any positive integers a and b.

(A.2) Lemma. For any three non-negative integers a, b and c, we have

$$\sum_{i=0}^{a} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} b \\ i+c \end{bmatrix} = \begin{bmatrix} a+b \\ a+c \end{bmatrix}.$$

PROOF. We use induction on a. If $a=0$ then the desired equality is trivial. We now assume that

$$\sum_{i=0}^{p} \begin{bmatrix} p \\ i \end{bmatrix} \begin{bmatrix} q \\ i+r \end{bmatrix} = \begin{bmatrix} p+q \\ p+r \end{bmatrix}$$

for any non-negative integers p, q and r such that $p < a$. Then using (A.1) we have

$$\sum_{i=0}^{a} \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} b \\ i+c \end{bmatrix} = \sum_{i=0}^{a-1} \begin{bmatrix} a-1 \\ i \end{bmatrix} \begin{bmatrix} b \\ i+c \end{bmatrix} + \sum_{i=1}^{a} \begin{bmatrix} a-1 \\ i-1 \end{bmatrix} \begin{bmatrix} b \\ i+c \end{bmatrix} = \begin{bmatrix} a-1+b \\ a-1+c \end{bmatrix} + \begin{bmatrix} a-1+b \\ a+c \end{bmatrix} = \begin{bmatrix} a+c \\ a+b \end{bmatrix}. \quad \text{q.e.d.}$$

(A.3) Lemma. For any non-negative integers a and for two non-negative integers b and c such that $c \geqq b$, we have

$$\sum_{i=0}^{b-1} (-1)^i \begin{bmatrix} a \\ i \end{bmatrix} \begin{bmatrix} c-i \\ b-i \end{bmatrix} = \begin{bmatrix} c-a \\ b \end{bmatrix} - (-1)^b \begin{bmatrix} a \\ b \end{bmatrix}.$$

PROOF. We use induction on a. First assume $a=1$. If $b=1$ then

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} c \\ 1 \end{bmatrix} = c = (c-1) - (-1) = \begin{bmatrix} c-1 \\ 1 \end{bmatrix} - (-1) \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

and if $b > 1$ then by (A.1)

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} c \\ b \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} c-1 \\ b-1 \end{bmatrix} = \begin{bmatrix} c-1 \\ b \end{bmatrix} = \begin{bmatrix} c-1 \\ b \end{bmatrix} - (-1)^b \begin{bmatrix} 1 \\ b \end{bmatrix}$$

because $\begin{bmatrix} 1 \\ b \end{bmatrix} = 0$ for $b > 1$. Thus we prove the case $a=1$. Now assume that, for any $p < a$ and for any b and c such that $c \geqq b \geqq 0$,

$$\sum_{i=0}^{b-1} (-1)^i \begin{bmatrix} p \\ i \end{bmatrix} \begin{bmatrix} c-i \\ b-i \end{bmatrix} = \begin{bmatrix} c-p \\ b \end{bmatrix} - (-1)^b \begin{bmatrix} p \\ b \end{bmatrix}.$$

Applying (A.1) for $\begin{bmatrix} a \\ i \end{bmatrix}$, we have

$$\begin{aligned}\sum_{i=0}^{b-1}(-1)^i\begin{bmatrix} a \\ i \end{bmatrix}\begin{bmatrix} c-i \\ b-i \end{bmatrix} &= \sum_{i=0}^{b-1}(-1)^i\begin{bmatrix} a-1 \\ i \end{bmatrix}\begin{bmatrix} c-i \\ b-i \end{bmatrix}+\sum_{i=1}^{b-1}(-1)^i\begin{bmatrix} a-1 \\ i-1 \end{bmatrix}\begin{bmatrix} c-i \\ b-i \end{bmatrix} \\ &= \sum_{i=0}^{b-1}(-1)^i\begin{bmatrix} a-1 \\ i \end{bmatrix}\begin{bmatrix} c-i \\ b-i \end{bmatrix}-\sum_{i=0}^{(b-1)-1}(-1)^i\begin{bmatrix} a-1 \\ i \end{bmatrix}\begin{bmatrix} (c-1)-i \\ (b-1)-i \end{bmatrix} \\ &= \begin{bmatrix} c-(a-1) \\ b \end{bmatrix}-(-1)^b\begin{bmatrix} a-1 \\ b \end{bmatrix}-\begin{bmatrix} (c-1)-(a-1) \\ b-1 \end{bmatrix}+(-1)^{b-1}\begin{bmatrix} a-1 \\ b-1 \end{bmatrix} \\ &= \begin{bmatrix} c-a \\ b \end{bmatrix}-(-1)^b\begin{bmatrix} a \\ b \end{bmatrix}.\end{aligned}$$

This prove Lemma (A.3) in general case.

(A.4) In formal power series ring $Z[[t]]$, coefficients of the Taylor expansion

$$(t+t^2+t^3+\cdots)^j=\sum_{i=j}^{\infty}a_{ij}t^i \quad (i\geqq j\geqq 1),$$

satisfy

$$a_{ij}=\begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \quad (i\geqq j\geqq 1).$$

Indeed,

$$(t+t^2+t^3+\cdots)^j=t^j\cdot(1-t)^{-j}$$

and

$$\frac{d^k}{dt^k}(1-t)^{-j}=j(j+1)\cdots(j+k-1)\cdot(1-t)^{-j-k+1}=\frac{(j+k-1)!}{(j-1)!}(1-t)^{-j-k+1}$$

thus we have

$$a_{ij}=\frac{1}{(i-j)!}\frac{d^{i-j}}{dt^{i-j}}(1-t)^{-j}\Big|_{t=0}=\frac{(i-1)!}{(i-j)!(j-1)!}=\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}. \quad \text{q.e.d.}$$

(A.5) Lemma. For any element $x\in K(X)$

$$\gamma^i(x)=\sum_{j=1}^{i}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}\lambda^j(x).$$

PROOF. Under the notation in (A.4), we can write

$$\gamma^i(x)=\sum_{j=1}^{i}a_{ij}\lambda^j(x)$$

for any $x\in K(X)$, because

$$\gamma_t(x)=\sum_{i=0}^{\infty}\gamma^i(x)\cdot t^i=\lambda_{t/(1-t)}(x)=\sum_{j=0}^{\infty}\lambda^j(x)(t+t^2+\cdots)^j=1+\sum_{j=1}^{\infty}\lambda^j(x)\sum_{i=j}^{\infty}a_{ij}t^i=1+\sum_{i=1}^{\infty}t^i\sum_{j=1}^{i}a_{ij}\lambda^j(x)$$

by the definition (1.6.1) of γ. Using (A.4) we prove (A.5).

(A.6) Proof of the equality (1.6.6) :

$$\gamma^i(cl(F)-r)=\sum_{j=0}^{i}(-1)^{i-j}\begin{bmatrix} r-j \\ i-j \end{bmatrix}\lambda^j(cl(F)).$$

We use induction on i. If $i=0$ then the equality is trivial. We may assume $i>0$. By the property (1.6.2(b)) of γ,

$$\gamma^{i}(\mathrm{cl}(F))=\sum_{k=0}^{i}\gamma^{k}(\mathrm{cl}(F)-r)\cdot\gamma^{i-k}(r)$$

hence

$$\gamma^{i}(\mathrm{cl}(F)-r)=\gamma^{i}(\mathrm{cl}(F))-\sum_{k=0}^{i-1}\gamma^{k}(\mathrm{cl}(F)-r)\cdot\gamma^{i-k}(r)$$

$$=\sum_{j=1}^{i}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}\lambda^{j}(\mathrm{cl}(F))-\sum_{k=0}^{i-1}\gamma^{k}(\mathrm{cl}(F)-r)\cdot\sum_{p=1}^{i-k}\begin{bmatrix} i-k-1 \\ p-1 \end{bmatrix}\lambda^{p}(r)$$

by Lemma (A.5). Since

$$\lambda^{p}(r)=\begin{bmatrix} r \\ p \end{bmatrix},$$

it follows

$$\sum_{p=1}^{i-k}\begin{bmatrix} i-k-1 \\ p-1 \end{bmatrix}\lambda^{p}(r)=\begin{bmatrix} (i-k-1)+r \\ (i-k-1)+1 \end{bmatrix}=\begin{bmatrix} i-k-1+r \\ i-k \end{bmatrix}$$

by applying Lemma (A.2) for $(a, b, c, i)=(i-k-1, r, 1, p-1)$. Thus using the induction hypothesis we have

$$\sum_{k=0}^{i-1}\gamma^{k}(\mathrm{cl}(F)-r)\cdot\sum_{p=1}^{i-k}\begin{bmatrix} i-k-1 \\ p-1 \end{bmatrix}\lambda^{p}(r)=\sum_{k=0}^{i-1}\sum_{q=0}^{k}(-1)^{k-q}\begin{bmatrix} r-q \\ k-q \end{bmatrix}\lambda^{q}(\mathrm{cl}(F))\begin{bmatrix} i-k-1+r \\ i-k \end{bmatrix}$$

$$=\sum_{q=0}^{i-1}\sum_{k=q}^{i-1}(-1)^{k-q}\begin{bmatrix} r-q \\ k-q \end{bmatrix}\begin{bmatrix} i-k-1+r \\ i-k \end{bmatrix}\lambda^{q}(\mathrm{cl}(F))$$

$$=\sum_{q=0}^{m}\sum_{k=q}^{i-1}(-1)^{k-q}\begin{bmatrix} r-q \\ k-q \end{bmatrix}\begin{bmatrix} i-k-1+r \\ i-k \end{bmatrix}\lambda^{q}(\mathrm{cl}(F))$$

$$=\sum_{q=0}^{m}\left(\begin{bmatrix} (i-q-1+r)-(r-q) \\ i-q \end{bmatrix}-(-1)^{i-q}\begin{bmatrix} r-q \\ i-q \end{bmatrix}\right)\cdot\lambda^{q}(\mathrm{cl}(F))$$

$$=\sum_{q=0}^{m}\left(\begin{bmatrix} i-1 \\ i-q \end{bmatrix}-(-1)^{i-q}\begin{bmatrix} r-q \\ i-q \end{bmatrix}\right)\cdot\lambda^{q}(\mathrm{cl}(F))$$

$$=\sum_{q=0}^{i-1}\left(\begin{bmatrix} i-1 \\ i-q \end{bmatrix}-(-1)^{i-q}\begin{bmatrix} r-q \\ i-q \end{bmatrix}\right)\cdot\lambda^{q}(\mathrm{cl}(F))$$

by Lemma (A.3) for $(r-q, i-q, i-q-1+r, k-q)$ insted of (a, b, c, i), where we set $m=\min\{i-1, r\}$ considering

$$\lambda^{q}(\mathrm{cl}(F))=0$$

for $q>r=\mathrm{rk}F$. Therefore we have

$$\gamma^{i}(\mathrm{cl}(F)-r)=\sum_{j=1}^{i}\begin{bmatrix} i-1 \\ j-1 \end{bmatrix}\lambda^{j}(\mathrm{cl}(F))-\sum_{k=0}^{i-1}\gamma^{k}(\mathrm{cl}(F)-r)\cdot\sum_{p=1}^{i-k}\begin{bmatrix} i-k-1 \\ p-1 \end{bmatrix}\lambda^{p}(r)$$

$$=\sum_{q=1}^{i}\begin{bmatrix} i-1 \\ q-1 \end{bmatrix}\lambda^{q}(\mathrm{cl}(F))-\sum_{q=0}^{i-1}\left(\begin{bmatrix} i-1 \\ q-1 \end{bmatrix}-(-1)^{i-q}\cdot\begin{bmatrix} r-q \\ i-q \end{bmatrix}\right)\lambda^{q}(\mathrm{cl}(F))$$

$$=\begin{bmatrix} i-1 \\ i-1 \end{bmatrix}\lambda^{i}(\mathrm{cl}(F))+\sum_{q=1}^{i-1}(-1)^{i-q}\begin{bmatrix} r-q \\ i-q \end{bmatrix}\lambda^{q}(\mathrm{cl}(F))+(-1)^{i}\begin{bmatrix} r \\ i \end{bmatrix}\lambda^{0}(\mathrm{cl}(F))$$

$$=\sum_{q=0}^{i}(-1)^{i-q}\begin{bmatrix} r-q \\ i-q \end{bmatrix}\lambda^{q}(\mathrm{cl}(F))$$

in $K(X)$. q.e.d.

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