SOME PROPERTIES OF TORUS EQUIVARIANT VECTOR BUNDLES II

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ABSTRACT

An n-dimensional projective space has a natural action of an n-dimensional algebraic torus over an algebraic closed field. The projective space becomes a toric variety with natural torus actions. We consider the vector bundle on the projective space which has equivariant actions of the torus. In this paper, we study cohomological dimensions of an indecomposable torus equivariant vector bundle of rank n.

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§ 1 Introduction

Let k be an algebraic closed field and let T be an n-dimensional algebraic torus. T acts naturally on an n-dimensional projective space P^n over k. Then P^n becomes a toric variety. We consider a vector bundle on an n-dimensional projective space P^n which has torus equivariant actions. We call this vector bundle a torus equivariant vector bundle. We can find the detail definition of a torus equivariant vector bundle in [3]. In this paper we study these torus equivariant vector bundles.

DEFINITION 1. Let $a_0,\ldots,a_n (n \geqq 2)$ be positive integers. We define a vector bundle $E = E(a_0,\ldots,a_n)$ of rank n on an n-dimensional projective space P^n by the exact sequence

$$0 \longrightarrow \mathrm{O}_{\mathrm{P}^n} \xrightarrow{f} \bigoplus_{i=0}^{n} \mathrm{O}_{\mathrm{P}^n}(a_i) \longrightarrow E(a_0,\ldots,a_n) \longrightarrow 0$$

such that f sends 1 to $(\mathrm{X}_0^{a_0},\ldots,\mathrm{X}_n^{a_n})$ where $\mathrm{X}_0,\ldots,\mathrm{X}_n$ are homogeneous coordinates of P^n.

We have the following theorem in [3] about indecomposable torus equivariant vector bundles of rank n on an n-dimensional projective space.

THEOREM 2. The indecomposable torus equivariant vector bundle of rank n $(n > 1)$ on an n-dimensional projective space is isomorphic to $E(a_0,\ldots,a_n)(k)$ or $E(a_0,\ldots,a_n)^*(k)$ for some positive integers $a_0,\dots,a_n$ and some integer k.

§ 2 Definitions and some results.

DEFINITION 3. We define a numerical function $g(k)$ for an integer k by

$$g(k) = \begin{cases} \frac{1}{n!}(k+1)(k+2)\cdots(k+n) & \text{if } k \geqq 0 \\ 0 & \text{if } k < 0. \end{cases}$$

DEFINITION 4. Let E be a vector bundle on an n-dimensional projective space P^n. Then we write the dimension of cohomology group $\mathrm{H}^i(\mathrm{P}^n, E)$ $(i \geqq 0)$ by $h^i(E) = \dim \mathrm{H}^i(\mathrm{P}^n, E)$.

PROPOSITION 5. Let $a_0,\dots,a_n$ be positive integers. Let $E = E(a_0,\dots,a_n)$ be a torus equivariant vector bundle of rank n $(n \geqq 2)$ on an n-dimensional projective space P^n. Then, for an integer k, we have

(1) $h^0(E^*(k)) = \sum_{i<j} g(k - a_i - a_j) - \sum_{i<j<l} g(k - a_i - a_j - a_l) + \sum_{i<j<l<m} g(k - a_i - a_j - a_l - a_m) - \dots$

(2) $h^0(E(k)) = \sum_i g(k + a_i) - g(k)$.

This proposition was proved in [5].

§ 3 Cohomological dimensions of torus equivariant vector bundles.

In this section we denote by $E = E(a_0,\dots,a_n)$ the torus equivariant vector bundle of rank n $(n \geqq 2)$ on an n-dimensional projective space $\mathrm{P} = \mathrm{P}^n$ where $a_0,\dots,a_n$ are positive integers. We may assume that $0 < a_0 \leqq \dots \leqq a_n$ by changing the projective coordinates. We denote $\mathrm{O}_\mathrm{P}(k)$ by $\mathrm{O}(k)$ for the convenience.

PROPOSITION 6. In the case of $n = 2$, for an integer k, we have the following formulae

(1) $h^1(E(k)) = g(-k-3) - \sum_i g(-k-3-a_i) + \sum_{i<j} g(-k-3-a_i-a_j) - \dots$

$h^1(E^*(k)) = g(k) - \sum_i g(k - a_i) + \sum_{i<j} g(k - a_i - a_j) - \dots$

(2) $h^2(E(k)) = \sum_{i<j} g(-k-3-a_i-a_j) - \sum_{i<j<l} g(-k-3-a_i-a_j-a_l) + \dots$

$$h^2(E^*(k)) = \sum_i g(-k-3+a_i) + g(-k-3).$$

Proof. We prove second assertions at first. By the duality theorem we have

$$H^2(P^2, E(k)) \simeq H^0(P^2, E^*(-k-3)) \quad \text{and} \quad H^2(P^2, E^*(k)) \simeq H^0(P^2, E(-k-3)).$$

Therefore, by virtue of Proposition 5, we have

$$\begin{aligned} h^2(E(k)) &= h^0(E^*(-k-3)) \\ &= \sum_{i<j} g(-k-3-a_i-a_j) - \sum_{i<j<l} g(-k-3-a_i-a_j-a_l) + \ldots \end{aligned}$$

and

$$\begin{aligned} h^2(E^*(k)) &= h^0(E(-k-3)) \\ &= \sum_i g(-k-3+a_i) + g(-k-3). \end{aligned}$$

Now we prove (1). From the exact sequence of the definition of the torus equivariant vector bundle E, we have an exact sequence

$$0 \longrightarrow O(k) \longrightarrow \bigoplus_i O(k+a_i) \longrightarrow E(k) \longrightarrow 0.$$

Then we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0(P^2, O(k)) \longrightarrow \bigoplus_i H^0(P^2, O(k+a_i)) \longrightarrow H^0(P^2, E(k)) \\ &\longrightarrow H^1(P^2, O(k)) \longrightarrow \bigoplus_i H^1(P^2, O(k+a_i)) \longrightarrow H^1(P^2, E(k)) \\ &\longrightarrow H^2(P^2, O(k)) \longrightarrow \bigoplus_i H^2(P^2, O(k+a_i)) \longrightarrow H^2(P^2, E(k)) \longrightarrow 0. \end{aligned}$$

Since $\bigoplus_i H^1(P^2, O(k+a_i)) = 0$ we have

$$\begin{aligned} h^1(E(k)) &= h^2(O(k)) - \sum_i h^2(O(k+a_i)) + h^2(E(k)) \\ &= g(-k-3) - \sum_i g(-k-3-a_i) + \sum_{i<j} g(-k-3-a_i-a_j) - \ldots \end{aligned}$$

This shows the first formula of (1). And furthermore from the dual exact sequence of the exact sequence of the definition of E, we have, by dualizing,

$$0 \longrightarrow E^*(k) \longrightarrow \bigoplus_i O(k-a_i) \longrightarrow O(k) \longrightarrow 0.$$

Then we have a long exact sequence

$$0 \longrightarrow H^0(P^2, E^*(k)) \longrightarrow \bigoplus_i H^0(P^2, O(k-a_i)) \longrightarrow H^0(P^2, O(k))$$

$$\longrightarrow H^1(P^2, E^*(k)) \longrightarrow \bigoplus_i H^1(P^2, O(k-a_i)) \longrightarrow H^1(P^2, O(k))$$

$$\longrightarrow H^2(P^2, E^*(k)) \longrightarrow \bigoplus_i H^2(P^2, O(k-a_i)) \longrightarrow H^2(P^2, O(k)) \longrightarrow 0.$$

Since $\bigoplus_i H^1(P^2, O(k-a_i)) = 0$ we have

$$h^1(E^*(k)) = h^0(O(k)) - \sum_i h^0(O(k-a_i)) + h^0(E^*(k))$$

$$= g(k) - \sum_i g(k-a_i) + \sum_{i<j} g(k-a_i-a_j) - \ldots$$

These complete the proof of Proposition 6.

PROPOSITION 7. In the case of $n=3$, for an integer k, we have the following formulae

(1) $h^1(E(k)) = 0$,

$$h^1(E^*(k)) = g(k) - \sum_i g(k-a_i) + \sum_{i<j} g(k-a_i-a_j) - \ldots$$

(2) $h^2(E(k)) = g(-k-4) - \sum_i g(-k-4-a_i) + \sum_{i<j} g(-k-4-a_i-a_j) - \ldots$

$h^2(E^*(k)) = 0$.

Proof. From the exact sequence of the definition of E we have

$$0 \longrightarrow O(k) \longrightarrow \bigoplus_i O(k+a_i) \longrightarrow E(k) \longrightarrow 0.$$

Then we have a long exact sequence. Since

$$\bigoplus_i H^1(P^3, O(k+a_i)) = 0 \quad \text{and} \quad H^2(P^3, O(k)) = 0$$

we have $h^1(E(k)) = 0$. On the other hand, from an exact sequence

$$0 \longrightarrow E^*(k) \longrightarrow \bigoplus_i O(k-a_i) \longrightarrow O(k) \longrightarrow 0.$$

we have a long exact sequence. Since $\bigoplus_i H^1(P^3, O(k-a_i)) = 0$ we have

$$h^1(E^*(k)) = h^0(O(k)) - \sum_i h^0(O(k-a_i)) + h^0(E^*(k))$$

$$= g(k) - \sum_i g(k-a_i) + \sum_{i<j} g(k-a_i-a_j) - \ldots$$

These proves (1). Now we will prove the second part of assertions. Using the duality theorem, we have

$$H^2(P^3, E(k)) \simeq H^1(P^3, E^*(-k-4)) \quad \text{and} \quad H^2(P^3, E^*(k)) \simeq H^1(P^3, E(-k-4)).$$

Therefore we have

$$h^2(E(k)) = g(-k-4) - \sum_i g(-k-4-a_i) + \sum_{i<j} g(-k-4-a_i-a_j) - \ldots$$

and

$$h^2(E^*(k)) = 0.$$

So we have proved Proposition 7.

PROPOSITION 8. In the case of $n > 3$, for an integer k, we have the following formulae

(1) $h^1(E(k)) = 0$,

$$h^1(E^*(k)) = g(k) - \sum_i g(k-a_i) + \sum_{i<j} g(k-a_i-a_j) - \ldots$$

(2) $h^2(E(k)) = 0$,

$$h^2(E^*(k)) = 0.$$

Proof. The two formulae of (1) are proved by the same reasons of the proof of the first part of Proposition 7. So we are going to prove the second part of proposition 8. We consider long exact sequences which are same as in the proof of Proposition 6. Since

$$\bigoplus_i H^2(P^n, O(k+a_i)) = 0 \quad \text{and} \quad H^3(P^n, O(k)) = 0.$$

we have $H^3(P^n, O(k)) = 0$. This means $h^2(E(k)) = 0$. Furthermore since

$$H^1(P^n, O(k)) = 0 \quad \text{and} \quad \bigoplus_i H^2(P^n, O(k-a_i)) = 0.$$

we have $H^3(P^n, E^*(k)) = 0$. This means $h^2(E^*(k)) = 0$. Hence the proposition was proved.

PROPOSITION 9. In the case of $n > 2$, we have

$$h^1(E^* \otimes E) - h^0(E^* \otimes E) = \sum_i g(a_i) - \sum_{i \leq j} g(a_j - a_i) - 1.$$

Proof. From the short exact sequence of the definition of E, we have, by tensoring,

$$0 \longrightarrow E^* \otimes E \longrightarrow \bigoplus_i E(-a_i) \longrightarrow E \longrightarrow 0.$$

Then we have a long exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathrm{P}, E^*\otimes E) \longrightarrow \bigoplus_i \mathrm{H}^0(\mathrm{P}, E(-a_i)) \longrightarrow \mathrm{H}^0(\mathrm{P}, E)$$

$$\longrightarrow \mathrm{H}^1(\mathrm{P}, E^*\otimes E) \longrightarrow \bigoplus_i \mathrm{H}^1(\mathrm{P}, E(-a_i)) \longrightarrow \mathrm{H}^1(\mathrm{P}, E) \longrightarrow \cdots$$

Since $h^1(E(-a_i))=0$ we have

$$h^1(E^*\otimes E) - h^0(E^*\otimes E) = h^0(E) - \sum_i h^0(E(-a_i))$$

$$= \sum_i g(a_i) - g(0) - \sum_{i,j} g(-a_i + a_j)$$

$$= \sum_i g(a_i) - \sum_{i \leqq j} g(a_j - a_i) - 1.$$

These complete the proof of the proposition.

§ 4 Isomorphisms between torus equivariant vector bundles.

In this section, we study the isomorphism between two torus equivariant vector bundles on an n-dimensional projective space P. Especially we consider the case such that $E(a_0,\ldots,a_n) \simeq E(b_0,\ldots,b_n)^*(l)$ where $a_0,\ldots,a_n,b_0,\ldots,b_n$ are positive integers and l is an integer. In this case we may assume that $0<a_0\leqq a_1\leqq\ldots\leqq a_n$, and $0<b_0\leqq b_1\leqq\ldots\leqq b_n$ by changing the coordinates of the projective space. In the paper[5], we had the following theorem.

THEOREM 10. In the case of $n=2$, assume that $E(a_0,a_1,a_2) \simeq E(b_0,b_1,b_2)^*(l)$ where $0<a_0\leqq a_1\leqq a_2$, and $0<b_0\leqq b_1\leqq b_2$ and l is an integer. Then we have

$$a_0=b_0,\ a_1=b_1,\ a_2=b_2, \text{ and } l=a_0+a_1+a_2.$$

THEOREM 11. In the case of $n\geqq 3$, There is no isomorphism between two torus equivariant vector bundles $E(a_0,\ldots,a_n)$ and $E(b_0,\ldots,b_n)^*(l)$ for any $0<a_0\leqq a_1\leqq\ldots\leqq a_n$, and $0<b_0\leqq b_1\leqq\ldots\leqq b_n$ and any integer l.

Proof. Assume that there is an isomorphism $E(a_0,\ldots,a_n) \simeq E(b_0,\ldots,b_n)^*(l)$ for some $0<a_0\leqq a_1\leqq\ldots\leqq a_n$, and $0<b_0\leqq b_1\leqq\ldots\leqq b_n$ and some integer l.
Put $E=E(a_0,\ldots,a_n)$ and $F=E(b_0,\ldots,b_n)$. Since $\mathrm{H}^1(E(k))=0$ for any integer k, by Proposition 7 and Proposition 8, we have $\mathrm{H}^1(F^*(l+k))=0$. From the short exact sequence of F, we have an exact sequence

$$0 \longrightarrow F^*(l+k) \longrightarrow \bigoplus_i \mathrm{O}(l+k-b_i) \longrightarrow \mathrm{O}(l+k) \longrightarrow 0.$$

Then, furthermore we have a long exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0(\mathrm{P}, F^*(l+k)) \longrightarrow \bigoplus_i \mathrm{H}^0(\mathrm{P}, \mathrm{O}(l+k-b_i)) \longrightarrow \mathrm{H}^0(\mathrm{P}, \mathrm{O}(l+k)) \\ &\longrightarrow \mathrm{H}^1(\mathrm{P}, F^*(l+k)) \longrightarrow \bigoplus_i \mathrm{H}^1(\mathrm{P}, \mathrm{O}(l+k-b_i)) \longrightarrow \mathrm{H}^1(\mathrm{P}, \mathrm{O}(l+k)) \longrightarrow \cdots \end{aligned}$$

Since $\bigoplus_i \mathrm{H}^1(\mathrm{P}, \mathrm{O}(l+k-b_i)) = 0$ we have

$$h^0(F^*(l+k)) = \sum_i h^0(\mathrm{O}(l+k-b_i)) + h^1(F^*(l+k)) - h^0(\mathrm{O}(l+k)).$$

Furthermore since $h^1(E(k))=0$ by Proposition 7 and Proposition 8, we have $h^1(F^*(l+k))=0$. Hence we have

$$h^0(F^*(l+k)) = \sum_i g(l+k-b_i) - g(l+k).$$

On the other hand, by Proposition 5, we have

$$h^0(F^*(l+k)) = \sum_{i<j} g(l+k-b_i-b_j) - \sum_{i<j<m} g(l+k-b_i-b_j-b_m) + \cdots$$

Therefore we have

$$g(l+k) - \sum_i g(l+k-b_i) + \sum_{i<j} g(l+k-b_i-b_j) - \cdots = 0$$

for any integer k. So we put $k = -l + b_0 - 1$ in the above equation. Then we have

$$g(b_0 - 1) = 0.$$

This means that $b_0 - 1 < 0$. This is a contradiction because b_0 is a positive integer. Hence there is no isomorphism between two torus equivariant vector bundles which are stated in the theorem.

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