Conductor ideals of Gorenstein domains and local rings with multiplicity 2

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Summary: it will be shown that the monomial conjecture is equivalent to the assertion that every complete Gorenstein domain has conductor property.

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Introduction.

In [Hu] Huneke proved the following theorem.

Theorem. Let A be a complete local ring containing a field, $e(A)$ the multipilicity of A ,and $n \geqq 2$ an integer. Suppose that A satisfies (S_n) and $e(A) \leqq n$. Then A is Cohen-Macaulay.

This theorem is deduced from the following facts:

(1) A contains a regular local subring R such that A is a finite module over R and $rank_R(A) = e(A)$

(2) R is a direct summand of A as R-module (Hochster's direct summand conjecture).

(3) Syzygy theorem of Evans-Griffith [EG].

Hochster [Ho2] showed that syzygy theorem for regular local rings is a consequence of the direct summand conjecture. O'carroll [O] showed that syzygy theorem for regular local rings holds if Huneke's theorem is true without the the assumption that A contains a field. We note that the property (1) does not always hold if A does not contain a field. For example, let $A = R[[X]]/(p^2 - X^3)$, where $R = Z_p$ is the ring of p-adic integaers. Clearly, $e(A) = 2$ and R is the only regular local subring of A over which A is a finite module. But, since A is Cohen-Macaulay, we have

$$rank_R(A) = l_R(A/pA) = 3 .$$

Huneke conjectures that his theorem holds without the assumption that A contains a field. It is easy to see that if A has property (1) then his conjecture is a consequence of the direct summand conjecture. For a complete local ring which does not contain a field, it is not known whether Huneke's conjecture is implied by the direct summand conjecture. Although Huneke stated that his conjecture is true for $n = 2$ no proof has been given and his conjecture seems to be open even for $n = 2$. The purpose of this paper to show that his conjecture for $n = 2$ is implied by the monomial conjecture.

In this paper a ring means a commutative Noetherian ring with unity and a local ring means a Noetherian local ring. (R,m) denotes a local ring R with maximal ideal m.

§ 1 A property of conductor ideals of Gorenstein domains.

We begin with recalling the monomial conjecture. (A,m) be a local ring with $dim\, A = d$ and $\underline{a} = a_1, a_2, \cdots, a_d$ a system of parameters of A.

Definition. We say that a system of parameters a of A has monomial property if

$$(a_1 a_2 \cdots ad)^n \notin (a_1^{n+1}, a_2^{n+1}, \cdots a_d^{n+1})$$

for all $n > 0$. We say that A has monomial property if every system of parameters of A has monomial property.

The monomial conjecture asserts that every local ring has monomial property [Ho$_1$]. We rephrase the monomial property in terms of local cohomology.
Let (A,m) be a local ring with $dim\, A = d$ and $\underline{a} = a_1, a_2, \cdots, a_d$ a system of parameters of A. $(\underline{a}^n)$ denotes the ideal of A generated by $a_1^n, \cdots, a_d^n$ for $n > 0$. Let $f_n : A/(\underline{a}^n) \rightarrow A/(\underline{a}^{n+1})$ be the homomorphism defined by $f_n(x\ mod\ (\underline{a}^n)) = a_1 a_2 \cdots a_d x$ mod $(\underline{a}^{n+1})$. Then $A/(\underline{a}^n)$, $n = 1,2,\cdots$ form a direct system and its limit is $H_m^d(A)$, the d-th local cohomology module of A [HK]. We have a canonical homomorphism

$$\phi(\underline{a}, 1) : A/(\underline{a}) \rightarrow H_m^d(A).$$

Then we see that $\underline{a}$ has monomial property if and only if $\phi(\underline{a},1)(1) \neq 0$. Note that the monomial conjecture holds if every complete normal local domain has monomial property, see [Ho1] or [Ho2].

Let (R,N) be a complete local Gorenstein domain and (A,m) a finite birational extension of R, and let $C = Ann_R(A/R)$ be the conductor ideal between A and R. The canonical module K_A is isomorphic to $Hom_R(A,R)$ by [HK]. It follows that K_A is isomorphic to C. By the multiplication we have a pairing

$$(*)\ C \otimes_R A \rightarrow R.$$

Let $\underline{a}$ be a system of parameters of R. Tensoring $(*)$ with R/a^nR, $n = 1,2,\cdots$, and taking the limit, we get a commutative diagram

$$\begin{array}{ccc} C \otimes_R A/\underline{a}^n A & \rightarrow & R/\underline{a}^n R \\ \downarrow & & \downarrow \eta_n \\ C \otimes_R H_m^d(A) & \underset{\psi}{\rightarrow} & H_N^d(R) , \end{array}$$

where η_n is the canonical homomorphism. ψ induces a homomorphism

$$\theta : H_m^d(A) \to Hom_R(C,H_N^d(R)).$$

Lemma 1. *θ is an isomorphism.*

Proof. We first note that,since R is Gorenstein, $H_N^d(R)$ is isomorphic to the injective envelope of R/N. Therefore,it is enough to show that the induced map

$$\theta' : C'' \to H_m^d(A)'$$

is an isomorphism,where $(\ )' = Hom_R(\ ,H_N^d(R))$. By Matlis duality, the canonical map $C \to C''$ is an isomorphism and composing it with θ' ,we get a map $\zeta : C \to H_m^d(A)'$ given by $\zeta(x)(\alpha) = \psi(x \otimes \alpha)$ for $x \in C$ and $\alpha \in H_m^d(A)$. On the other hand,we have a canonical isomorphism

$$H_m^d(A) \to H_N^d(R) \otimes_R A.$$

By this map,an element $\alpha \in H_m^d(A)$ represented by $(y \bmod \underline{a}^n A) \in A/\underline{a}^n A$ is mapped to $\eta_n(1) \otimes y$ and we have $\zeta(x)(\alpha) = \eta_n(xy)$,where $\eta_n : R/a^n R \to H_N^d(R)$ is the caononical homomorphism. Hence we get

$$\begin{aligned} C &\to Hom_R(H_m^d(A),H_N^d(R)) \\ &\simeq Hom_R(H_N^d(R)\otimes_R A,H_N^d(R)) \\ &\simeq Hom_R(A,R) \end{aligned}$$

by Matlis duality. It is not hard to see that the image of $x \in C$ in $Hom_R(A,R)$ is the multiplication by x and hence $C \to Hom_R(A,R)$ is an isomorphism and it follows that θ is also an isomorphism.

Prpopositon 2. *Let R be a complete Gorenstein local domain, A a finite birational extension of R and $C = Hom_R(A,R)$. Suppose that A has monomial property. Then for any parameter ideal q of R we have $C \not\subset q$.*

Proof. Let $q =(a_1,a_2,\cdots,a_d)$ and $\underline{a} = a_1 ,a_2 ,\cdots, a_d$. We use the same notation as in Lemma 1. By Lemma 1 and the monomial property of A we have $\alpha := \phi(\underline{a},\dot{1})(1) \neq 0$. Hence there exists an element $x \in C$ such that

$$\theta(x \otimes \alpha) = \eta_1(x+\underline{a}R) \neq 0.$$

Therefore $x \notin q$ and hence $C \not\subset q$.

Definition. We say that a local domain R has conductor property if $Ann_R(R^*/R) \not\subset q$ for any parameter ideal of R ,where R^* is the integral closure of R in its quotient field.

By Proposition 2 we see that if the monomial conjecture holds then every complete Gorenstein domain has conductor property. We can show that the converse also holds.

Theorem 3. *The following statements are equivalent.*
(1) The monomial conjecture holds.
(2) Every complete Gorenstein local domain has conductor property.

To prove this we need a lemma.

Lemma 4. *Let A be a complete local domain and let $a_1, a_2, \cdots, a_d$ be a system of paeameters of A. Then there is a subring R of A such that*
(1) R is a complete Gorenstein local domain,
(2) A is a finite birational extension of R and
(3) $a_1, a_2, \cdots, a_d \in R$.

Proof. Let S be a coefficient ring of A and let $R_1 = S[[a_1, a_2, \cdots, a_d]]$. Then A is finite over R_1 and R_1 is a regular local ring or a hypersurface. Let $Q(A)$ and $Q(R_1)$ be hte quotient fields of A and R_1, respectively. If $[Q(A) : Q(R_1)] = 1$ then R_1 fulfills the requirements. Suppose that $[Q(A) : Q(R_1)] > 1$. Let n be the smallest natural number for which there is a monic polynomial $f(X) \in R_1[X]$ with $deg\, f(X) = n$ and $f(x) = 0$ for some $x \in A - Q(R_1)$. We claim that

$$R_1[x] = R_1[X]/f(X)R_1[X].$$

Suppose that there is a polynomial $g(X) = a_m X^m + \cdots + a_0 \in R_1[X]$ such that $a_m \neq 0,\ 2 \leqq m < n$, and $f(x) = 0$. Then the monic polynomial

$$h(X) = X^m + a_m a_{m-1} X^{m-1} + \cdots + a_m^{m-1} a_0$$

over R_1 satisfies $h(a_m x) = 0$ and we have $a_m x \in A - Q(R_1)$. This contradicts to the minimality of $deg\, f(X)$. This proves the claim and $R_1[x]$ is a complete intersection. Since $[Q(A) : Q(R1[x]) < [Q(A) : Q(R_1)]$, the proof will be completed by induction.

Proof of Theorem 3. It is enough to show the implication $(2) \Rightarrow (1)$.
As remarked before it is enough to show that every complete normal local domain has monomial property. Let A be a complete normal local domain and $a_1, a_2, \cdots, a_d$ a system of parameters of A. Take a Gorenstein local ring R as in Lemma 4 which contains $a_1, a_2, \cdots, a_d$. Suppose that the monomial conjecture fails to hold for A and $a_1, a_2, \cdots, a_d$. Then there is a natural number n for which there exist $x_1, x_2, \cdots, x_d \in A$ such that

$$(*)\quad (a_1 a_2 \cdots a_d)^n = x_1 a_1^{n+1} + x_2 a_2^{n+1} + \cdots + x_d a_d^{n+1}.$$

By (2) there is an element $x \in Ann_R(A/R) - (a_1, a_2, \cdots, a_d)R$. From (2) we have

$$x(a_1a_2\cdots a_d)^n = (xx_1)a_1^{n+1} + (xx_2)a_2^{n+1} + \cdots + (xx_d)a_d^{n+1}.$$

But,this implies that $x \in (a_1, a_2, \cdots, a_d)R$,contradicting to the choice of x. Therefore,the monomial conjecture holds.

Hochster [Ho_1] showed that every local ring containg a field has monmial property.

Corollary 5. Every complete Gorenstein local domain containing a field has conductor property.

Remark. Let (R,m) be a local domain of dimension 1 . If the normalization R^* of R is finite over R then R has conductor property. In fact, if $C = Ann_R(R^*/R) \subseteq aR$ for some $a \in m$ we can write $C = aJ$ for some ideal of R .Then we have $JR^* = (1/a)CR^* = (1/a)C = J$ .This shows that $J \subseteq C$ and $C \subseteq aC$. This implies that $C = (0)$ by Nakayama's lemma . This is impossible since R^* is finite over R . A local ring A is said to be quasi-Gorenstein if the canonical module K_A is isomorphic to A . If A is a complete quasi-Gorenstein domain containing a field A has conductor property.To see this, let B be the normalization of A and $\underline{a}$ a system of parameters of A . Take a Gorenstein subring R of A as in Lemma 4 . Then we have $C_{B/R} = Ann_R(B/R) \subseteq C_{A/R} = Ann_R(A/R) \simeq K_A$. Since A is quasi-Gorenstein we have $C_{B/R} \subseteq aA$ for some $a \in C_{A/R}$ and we have $C_{B/R} = aJ$ for some ideal J of A . Then we get $C_{B/R} = BC_{B/R} = aJB = aJ$,which implies $JB = J$. Therefore, $J \subseteq C_{B/A}$ and consequently,we have $C_{B/R} \subseteq C_{B/A}C_{A/R}$. Suppose that $C_{B/A} \subseteq \underline{a}A$. Then we have $C_{B/R} \subseteq \underline{a}AC_{A/R} \subseteq \underline{a}R$. This contradicts to Corollary 5 . Hence $C_{B/A} \not\subseteq \underline{a}A$.

§ 2 Local rings with multiplicity 2 .

In this section we study local rings with multiplicity 2 satisfying (S_2). The conjecture is that such rings are Cohen-Macaulay. We first show that if the local ring is not a domain the conjecture is true.

Proposition 6. Let R be a complete local ring with $e(R) = 2$ .Suppose that R is not a domain. Then R is Cohen-Macaulay if and only if R satisfies (S_2) .

Proof. Since R is catenary and satisfies (S_2) we see that R is unmixed by [G2],(5.10.9). Hence,by the associativity of multiplicity we get

$$2 = \sum_{p \in Ass(R)} l(R_p)e(R/p).$$

There are two possibilities.

Case 1. $Ass(R) = \{ p_1, p_2 \}$

In this case R/p_1 and R/p_2 are regular by [N],(40.6). From the exact sequence

$$(*)\quad 0 \to R \to R/p_1 \oplus R/p_2 \to R/p_1 + p_2 \to 0$$

we see that for any $Q \in Ass(R/p_1 + p_2)$ we have $ht(Q) = 1$ by (S_2)-condition on R. Hence $ht(Q/p_1) = 1$ for all $Q \in Ass(R/p_1 + p_2)$.

Therefore, $(p_1 + p_2)/p_1$ is a height 1 unmixed ideal of the regular local ring R/p_1. Hence, $R/p_1 + p_2$ is a Cohen-Macaulay ring with dimension $d - 1$. The exact sequence (*) shows that R is Cohen-Macaulay.

Case 2. $Ass(R) = \{ p \}$.

In this case $e(R/p) = 1$ and $l(R_p) = 2$. This implies that $p^2 = (0)$ and R/p is regular. The exact sequence

$$(**)\quad 0 \to p \to R \to R/p \to 0$$

shows that p is a reflexive R/p-module of *rank 1*. Hence p is free as an R/p-module. From (**) we see that R is Cohen-Macaulay.

We are going to show that if a complete local domain has the property

(1) in the introduction then the conjecture is true.

The idea of the following proof is due to Hochster.

Proposition 7. Let (R,m) is a complete local ring with $e(R) = 2$. Suppose that R contains a regular local subring (S,n) such that R is a finite S-module and $e(R) = rank_S(R)$. Then R is Cohen-Macaulay if R satisfies (S_2).

Proof. By induction on $d = dim\, R$. The assertion is clear if $d \leqq 2$. Suppose that $d \geqq 3$. Let $p \in Spec(S)$ be a prime ideal of S which is not maximal. We claim that R_p is Cohen-Macaulay. Let T be the completion of S_p and A the pS_p-adic completion of R_p. Then A is either a product of two complete local rings,say $A = A_1 \times A_2$ or a complete local ring. Since R is complete A satisfies (S_2). Then we get

$$2 = e(R) = rank_S(R) = rank_T(A).$$

If $A = A_1 \times A_2$ then both A_1 and A_2 satisfy (S_2) and $e(A_1) = e(A_2) = 1$ because $rank_T(A_i) \geqq e(A_i)$. Hence A_i is regular by [N],(40.6) and consequently R_p is Cohen-Macaulay. Suppose that A is a local ring. Then R_p is a local ring and A is the completion of R_p. By [N],(40.1) we get

$$2 = e(R) \geqq e(R_p).$$

If $e(R_p) = 1$ then R_p is regular. Suppose that $e(R_p) = 2$. Then $rank_T(A) = e(A) = 2$.Then by induction hypothesis A is Cohen-Macaulay. Therefore R_p is Cohen-Macaulay ,as claimed. Now the claim implies that the exact sequence

$$0 \to S_p \to R_p \to (R/S)_p \to 0$$

is split exact and $(R/S)_p$ is Cohen-Macaulay for any $p \in Spec(S) - \{ n \}$. Since $depth\ R/S = 2$ we see that R/S is a reflexive S-module of rank 1 . Therefore R/S is a free S-module. Hence R is Cohen-Macaulay.

Proposition 8. *Let (R,m) be a complete Gorenstein domain with $e(R) = 2$ and let A be a finite birational extension of R . Suppose that A satisfies (S_2) . If the conductor $C_{A/R}$ is not contained in a minimal reduction $q = (a_1, a_2, \cdots, a_d)R$ of m with $d = \dim R$. Then A is Cohen-Macaulay.*

Proof. Recall that $C_{A/R}$ is isomorphic to the canonical mudule K_A of A and $C_{A/R}$ is a height 1 unmixed ideal of R ,(see [HK]). Since A satisfies (S_2) , the canonical map $A \to Hom_R(Hom_R(A,R),R)$ is an isomorphism. On the other hand, $C_{A/R}$ is isomorphic to $Hom_R(A,R)$. Hence A is isomorphic to $Hom_R(C_{A/R}, R)$. By local duality it is enough to show that $C_{A/R}$ is Cohen-Macaulay as R-module. Clearly, we may assume that $A \neq R$. By assumption $l_R(R/q) = 2$ and $C_{A/R} + q/q \neq (0)$. We have $C_{A/R} + q = m$. Since $a_1, a_2, \cdots, a_d$ can be extended to a minimal generating set of m , $C_{A/R}$ contains $v - d$ elements of a minimal generating set of m ,where v is the embedding dimension of R .Hence the embedding dimension of $R/C_{A/R}$ is at most d and $\dim R/C_{A/R} = d - 1$. Since $C_{A/R}$ is unmixed $R/C_{A/R}$ is a hypersurface. Now we can conclude that $C_{A/R}$ is Cohen-Macaulay.

As a corollary of Proposition 8 we have:

Corollary 9. *If the monomial conjecture holds then Huneke's conjecture for $n = 2$ is true.*

Question: Under which condition a local domain has conductor property ?

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