Quasi-Gorenstein homogeneous Buchsbaum rings with multiplicity 4

Shin Ikeda

summary

Quasi-Gorenstein homogeneous Buchsbaum rings with multiplicity 4 are classified. They occupy only one isomorphism class over a field of characteristic 2.

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Introduction

Let k be a field of characteristic 2 and $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$, be indeterminates over k. We put $R=k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]/J$, where J is the ideal generated by $X_1Y_1+X_2Y_2+X_3Y_3$, Y_1^2, Y_2^2, Y_3^2, Y_4^2, Y_1Y_4, Y_2Y_4, Y_3Y_4, $Y_1Y_2+X_3Y_4$, $Y_2Y_3+X_1Y_4$, $Y_1Y_3+X_2Y_4$. Then R is a non Cohen-Macaulay Buchsbaum ring with $K_R=R(-2)$, cf. [I_2]. Let M be the maximal homogeneous ideal of R and let $R(M)=\bigoplus_{n\geq 0} M^n$ be the Rees algebra of M. By [I_2] we know that $R(M)$ is a Gorenstein ring The purpose of this paper is to classify non-Cohen-Macaulay homogeneous graded Buchsbaum rings S over an algebraically closed field k such that $e(S)=4$ and $k_S=S(n)$ for some integer n. The conclusion of the classification is that S is isomorphic to the ring R at the beginning of this introduction.

For the classification we need to classify the homogeneous graded Buchsbaum ring R with $e(R)=3$, which satisfies (S_2). It turns out that if the base field k is algebraically closed and if R is not Cohen-Macaulay then R is isomorphic to $k[X_1, X_2, X_3, Y_1, Y_2, Y_3]/(X_1Y_1+X_2Y_2+X_3Y_3, (Y_1, Y_2, Y_3)^2)$ as k-algebra.

1 Preliminaries

Let k be a field and let $R=\bigoplus_{n\geq 0} R_n$ be a graded ring. R is called defined over k if $R_0=k$ and if R is a Noetherian ring. A graded ring R defined over k is said to be homogeneous if R is generated by homogeneous elements of degree 1 as a k-algenbra,i.e $R=k[R_1]$. Let R be a graded ring defined over k and $d=\dim R$. Homogeneous elements $x_1, x_2,\cdots$,

x_d are called a homogeneous system of paprameters of R if dim $R/(x_1,\cdots, x_d)R=0$. Let $S=k[x_1\cdots, x_d]$ be a subring of R generated by $x_1,\cdots,x_d$. Then S is isomorphic to a polynoimial ring in d variables over k and R is a finitely generated S-module. Let $K_R=Hom_S(R, S)$. K_R is called the canonical module of R, cf [GW]. By [GW] we know that K_R is independent of the choice of $x_1,\cdots,x_d$. It is easy to see that K_R is reflexive S-module. $E=Hom_R(K_R,K_R)$ will be called the S_2-fication of R. The kernel of the canonical map $R\to E$ is the ideal $U=\cap_{p\in \mathrm{Assh}(R)}\mathrm{Ker}(R\to R_p)$, where Assh($R$) denotes the set of prime ideals p of R such that dim $R=$dim $R/$p. We note that E is a subring of the total quotient ring of R/U. The cokernel of $R\to E$ has dimension at most d-2,cf.[HK]. A graded ring defined over k is called Buchsbaum if R_M is a Buchsbaum local ring, where M is the maximai homogeneous ideal of R. An ideal I of R is said to be unmixed if R/I is equidimensional. In this paper e(R) denotes the muitiplicity e(R_M) of the local ring R_M and v(R) denotes the embedding dimension of R_M. Let L be a finitely generated R -module. We say that L satisfies (S_2) if depth $L_P \geqq$ min $\{2,\mathrm{ht}(p)\}$. We first recall a simple property of a ring with (S_2).

Lemma 1. Let R be a Noetherian ring which satisfies (S_2). *If R is catenary then R is equidimensional.*

Proof. See [G],(5.10.9). Q.E.D.

Lemma 2. *Let R be a graded ring defined over a field k and let I and J be unmixed ideals of height* 0 *of R with* $I\cap J=(0)$ *and* $ht(I+J)>0$. *Suppose that R satisfies* (S_2). *Then* $I+J$ *is unmixed of height* 1. *If, moreover, R is homogeneous and* $e(R/I)=1$ *then* $R/I+J$ *is a hypersurface.*

Proof. Let $Q \in$ Ass($R/I+J$). From the exact sequence

$$0\to R_Q\to(R/I)_Q \oplus (R/J)_Q\to(R/I+J)_Q\to 0$$

we see that depth $R_Q=1$. Since R satisfies (S_2) we have height $Q=1$. The second assertion follows from the fact that R/I is a polynomial ring over k and $I+J/I$ is an unmixed ideal of height 1 of R/I. Q.E.D.

Lemma 3. *Let R be a graded ring defined over a field k.*
Suppose that R satisfied (S_2) *and* e(R)$\leqq 2$. *Then R is a hypersurface.*

Proof. See [H]. Q.E.D.

Lemma 4. *Let R be a reduced homogeneous ring over an algebraically* closed field k with e (R)$=2$. *Suppose that R is equidimensional. Then either*

(1) *R is Cohen-Macaulay or*

(2) *there are associated primes* p_1 *and* p_2 *of* R *such that* $ht(p_1+p_2)\geqq 2$.

Proof. Let E be the S_2-fication of R. Since E is a finitely generated R-module we have $E_n=(0)$ for $n\ll 0$. Suppse that E is not a product of graded rings. Let $x\in E_n$ and $n<0$. Then $x^m=0$. Then $x^m=0$ for some $m\geqq 1$. Since E is reduced $x=0$. If $y\in E_o$ y is algebraic over k Hence $x\in k$ since k is algebraically closed. By Lemma 3, E is a hypersurface. Hence we get $\dim_K R_1\leqq \dim_k E_1\leqq \dim R+1$. Hence R is a hypersurface and therefore R is Cohen-Macaulay. If E is a product of graded rings,say $E=E_1\times E_2$. Let $x_1,\cdots,x_d$ be a homogeneous system of parameters of degree 1 and set $S=k[x_1,x_2\cdots,x_d]$. *Then* $e(R)=\mathrm{rank}_S(R)=\mathrm{rank}_S(E)$. Hence $\mathrm{rank}_S(E_1)=\mathrm{rank}_S(E_2)=1$. Since E_i is a reflexive S-module E_i is free as S-module and hence E_i is Cohen-Macaulay. It follows that $\dim_k E_i/xE_i=1$, where $x=x_1,\cdots,x_d$. Therefore $E_i=R/p_i$ for some $p_i\in \mathrm{Ass}(R)$. Let C be the cokernel of the canonical map $R\to E$.

From the exact sequence $\qquad 0\to R\to R/p_1\times R/p_2\to C\to 0 \qquad (*)$

we see that $\dim_k C/xC=1$ and hence $C=R/q$ for some homogeneous ideal q of R. So, one gets $q=p_1+p_2$. Since $\dim C\leqq \dim R$-2 we have $\mathrm{ht}(p_1-p_2)\geqq 2$. Q.E.D.

Now let us recall a property of a homogeneous ring over an algebraically closed field.

Lemma 5. Let R be a homogeneous graded domain over an algebraically closed field k. Then $v(R)\leqq e(R)+\dim R$-*1.*

Proof. See, for example, [A] , (12.3.5). Q.E.D.

Corollary 6. Let R be as in Lemma 5. *If* $e(R)=2$ *then* R *is a hypersurface.*

Proof. Obvious. Q.E.D.

The following result plays a pivotal role in the sequel.

Theorem 7. *Let R be a homogeneous Buchsbaum ring over an algebraically closed field k. Suppose that R satisfies* (S_2) *and* $e(R)=3$. *If R is not Cohen-Macaulay then R is isomorphic to*

$$k[X_1, X_2, X_3, Y_1, Y_2, Y_3]/(X_1Y_1+X_2Y_2+X_3Y_3,(Y_1, Y_2, Y_3)^2),$$

as k-algebra, where $X_1, X_2, X_3, Y_1, Y_2, Y_3$, *are variables over k.*

Proof. By $[G_2]$, Theorem (4.1) we have

$$3=e(R)\geqq 1+\sum_{i-1}^{d-1}\binom{d-1}{i-1}h^i(R),$$

where $h^i(R)$ denotes the length of the local cohomology $H^i_M(R)$ with respect to the maximal ideal M of R and $d=\dim R$. Since R is not Cohen-Macaulay there is an integer

$i \geqq 2$ such that $h^i(R) \neq 0$. Hence we see that $d=3$ and $h^2(R)=1$. By [G_2], Theorem(4.12) R_M is a Buchsbaum ring of maximal embedding dimension. Let $x_1, \cdots, x_d$ be a homegeneous system of parameters of R and $S=k[x_1, x_2, \cdots, x_d]$. Also by [$G_2$] we have $M^2=(x_1 \cdots x_d)M$. Now we claim that R is not a domain. In fact, if R is a domain we have

$$e(R)+d-1 \geqq v(R)=e(R)+d+I(R)-1$$

by Lemma5. Therefore we get $I(R)=0$, which means that R is Cohen-Macaulay. This contradicts to the assumption,

Claim: R has only one associated prime.

The claim follows from the following lemma.

Lemma 8. *Let R be a homogeneous ring over an algebraically closed fieid k with dim $R=e(R)=3$. Suppose that R has at least two associated prime ideals. Then R is Cohen -Macaulay if and only if R satisfies* (S_2).

Proof. Suppose that R satisfies (S_2).

Case 1: $Ass(R)=\{p_1, p_2\}$ and R is reduced.

One may assume that $e(R/p_1)=1$ and $e(R/p_2)=2$. By Corollary 6 R/p_2 is Cohen -Macaulay. From the exact sequence

$$0 \to R \to R/p_1 \oplus R/p_2 \to R/p_1+p_2 \to 0$$

we see that R is Cohen-Macaulay.

Case 2.: $Ass(R)=\{p_1, p_2\}$ and R is not reduced.

In this case we may assume that $e(R/p_1)=e(R/p_2)=1$. Let N be the nilradical of R. Then N is an unmixed ideal with $ht(N)=0$ and $e(R/N)=2$. By Lemma 2 we have ht $(p_1+p_2)=1$ and hence R/N is Cohen-Macaulay by Lemma 1 and Lemma 4. From the exact sequence

$$0 \to N \to R \to R/N \to 0 \qquad (**)$$

we see that N satisfies (S_2). Let S be a subring of R generated by a homogeneous system of parameters of degree.1. Then N is reflexive of rank 1 as S-module. Therefore N is a free S-module. Hence R is Cohen-Macaulay by $(**)$.

Case 3 : $Ass(R)=\{p_1, p_2, p_3\}$.

In this case $e(R/p_i)=1$ for $i=1,2,3$. For any $i \neq j$, $p_i \cap p_j$ is Cohen-Macaulay R-module, since $e(R/p_1 \cap p_2)=2$. Hence, if $R/p_i \cap p_j$ is Cohen-Macaulay it follows that R is Cohen -Macaulay. Therefore we may assume that $R/p_i \cap p_j$ is not Cohen-Macaualy for $i \neq j$. By Lemma 4 we have

$ht(p_i+p_j) \geqq 2$ for $i \neq j$. Let $I=p_1 \cap p_2+p_1 \cap p_3$. Then p_1 is the unique minimal prime ideal of I and $e(R/I)=1$. We will show that $I=p_1$. Let $Q \in Ass(R/I)$. If $ht(Q) \geqq 2$, from the exact sequence

$$(\#) \qquad 0 \to R \to R/p_1 \cap p_2 \oplus R/p_1 \cap p_3 \to R/I \to 0,$$

we get that depth $R_Q = 1$, which contradicts (S_2)-condition on R.
If $\mathrm{ht}(Q) = 1$ we see that Q does not contain $p_1 + p_2$, because $\mathrm{ht}(p_1 + p_2) \geqq 2$. Therefore $Q \supset p_1$ and this implies Q contains neither p_2 nor p_3.
Hence we get

$$\mathrm{depth}\ (R/I)_Q = \mathrm{depth}(R/p_1)_Q > 0.$$

But this is impossible, because $Q \in \mathrm{Ass}(R/I)$. Therefore $\mathrm{ht}(Q) = 0$, which implies that $Q = p_1$.
Hence, by the exact sequence (#), we see that $R/p_1 \cap p_2$ and $R/p_1 \cap p_3$ satisfy (S_2). By Lemma 3 they are Cohen-Macaulay. The exact sequence (#) shows that R is Cohen-Macaulay.

Let us return to the main proof. Let p be the unique associated prime of R. Then we have $e(R) = l(R_p)e(R/p)$, where $l(R_p)$ is the length of the artinian ring R_p. Since R is not a domain we have $e(R/p) = 1$ and $l(R_p) = 3$. By [G_1] we have $v(R) = e(R) + \dim R + I(R) - 1 = 6$. Therefore we may assume that $R = k[x_1, x_2, x_3, y_1, y_2, y_3]$ and $p = (y_1, y_2, y_3)$ and also we may assume that x_1, x_2, x_3 form a homogeneous system of parameters of degree 1 of R. Since $M^2 = (x_1, x_2, x_3)M$ we see that R is generated by $1, y_1, y_2, y_3$ as S-module, where $S = k[x_1, x_2, x_3]$. Hence we have

$$R = S + Sy_1 + Sy_2 + Sy_3 = S + p.$$

Since $S \cap p = (0)$ we have $R = S \oplus p$ as S-module. Since R satisfies (S_2) R is a reflexive S-module and hence so is p with $\mathrm{rank}_S(p) = 2$. Therefoe we have a finite free resolution of p as S-module

$$0 \to S(-n) \xrightarrow{f} S^3(-1) \xrightarrow{g} p \to 0,$$

where f is given by a matrix $[f_1, f_2, f_3]$ for some homogeneous element f_i of S and n is an integer. This induces an isomorphism

$$Ext_S{}^1(p,S) \xrightarrow{\sim} S/(f_1, f_2, f_3)(n).$$

On the other hand

$$Ext_S{}^1(R,S) \xrightarrow{\sim} Ext_S{}^1(S,S) \oplus Ext_S{}^1(p,S) \xrightarrow{\sim} Ext_S{}^1(p,S).$$

By the duality of graded rings [GW] we know that $l_R(EXt_S{}^1(R,S)) = 1$, since $h^2(R) =$

1. Therefore $S/(f_1, f_2, f_3)\tilde{\to}k$. We may assume that $f_i=x_i$ for $i=1, 2, 3$ and $x_1y_1+x_2y_2+x_3y_3=0$.

Let $A=k[X_1, X_2, X_3, Y_1, Y_2, Y_3]$ be a polynomial ring and let $\phi : A\to R$ be a homomorphism given by $\phi(X_i)=x_i$ and $\phi(Y_i)=y_i$. Then $\ker\phi$ is generated by 7 homogeneous elements of degree 2 by [G_1], Theorem (1.1) and Theorem (4.4). If we can show that $p^2=(0)$ the proof will be completed, because we get $x_1y_1+x_2y_2=x_3y_3=0$, $y_1^2=y_1y_2=y_1y_3=y_2^2=y_2y_3=y_3^2=0$.

Let $p^{(2)}$ be the second symbolic power of p. If $p^{(2)}=(0)$ there is nothing to prove. Suppose that $p^{(2)}\neq(0)$. Then $p^{(2)}$ a reflexive S-module of rank 1. Hence $p^{(2)}$ is a principal ideal of R. Let $p^{(2)}=fR$.

We may assume that f is a homogeneous element of degree 1 or 2.

Case 1 : $\deg f=1$.

In this case we may assume that $f=y_1$. Since $y_1p\subset p^{(3)}=(0)$ we have $y_1^2=y_1y_2=y_1y_3=0$. Moreover, we get

$$y_2^2=y_1a$$
$$y_2y_3=y_1b$$
$$y_3^2=y_1c$$

for some $a,b,c\in S_1$. From the relation $x_1y_1+x_2y_2+x_3y_3=0$ we get

$$0=x_1y_1y_2+x_2y_2^2+x_3y_2y_3=ax_2y_1+bx_3y_1.$$

Hence

$$ax_2+bx_3\in S\cap p=(0).$$

Similarly we have $bx_2+cx_3=0$. Since x_2, x_3 is a regular sequence of S

$$b\in(x_1)\cap(x_2)=(x_1x_2).$$

Therefore $b=0$ since $\deg b=1$. It follows that $a=c=0$.

Case 2 : $\deg f=2$.

We can write

$$y_iy_j=\alpha_{ij}f$$

for some $\alpha_{ij}\in k$. Again from the relation $x_1y_1+x_2y_2+x_3y_3=0$ we get

$$(\alpha_{11}x_1+\alpha_{12}x_2+\alpha_{13}x_3)f=0,$$

which implies

$$\alpha_{11}x_1+\alpha_{12}x_2+\alpha_{13}x_3=0.$$

Hence $\alpha_{11}=\alpha_{12}=\alpha_{13}=0$.

Similarly we can show that every $\alpha_{ij}=0$.

We have completed the proof. Q.E.D.

Remark 9. If R is not homogeneous Theorem 7 is false. For example, let $X_1, X_2, X_3, Y_1, Y_2, Y_3$ be indeterminates over a field k with geg $X_i=1$ and deg $Y_i=2$ and let

$$R=k[X_1, X_2, X_3, Y_1, Y_2, Y_3]/I,$$

where I is the ideal generated by $X_1Y_1+X_2Y_2+X_3Y_3$, Y_1^2, Y_1Y_2, Y_1Y_3, $Y_2^2-X_3^2Y_1$, $Y_2Y_3+X_2X_3Y_1, Y_3^2-X_3^2Y_1$. Then R is a graded ring but not homogeneous over k. It is not hard to show that $R(MR_M)$ is Cohen-Macaulay by [HIO] , (48.23), where M is the maximal homogeneous ideal of R.

By $[I_1]$ R is Buchsbaum and satisfies (S_2). Since $M^2=(x_1, x_2, x_3)M$, where x_i is the image of X_i in R, we get $6=v(R)=e(R)+\dim R+I(R)-1$. It is easy to see that dim $R=3$ and $I(R)=l_R((x_1, x_2) : x_3/(x_1, x_2))=1$. Hence $e(R)=3$.

2 Quasi-Gorenstin homegeneous Buchsbaum rings.

A Noetherian local ring (A,m,k) is called quasi-Gorenstein if the canonical module K_A is isomorphic to A. A graded ring R over a field k is called quasi Gorenstein if R_M is quasi-Gorenstein, where M is the maximal homogeneous ideal of R.

Theorem 10. *Let R be a quasi-Gorenstein homogeneous Buchsbaum ring over an algebraically closed field k with maximal homogeneous ideal M. If* $e(R)=4$ *and R is not Cohen-Macaulay then* $ch(k)=2$ *and* R is isomorphic to

$$k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]/I,$$

where I is the ideal generated by $X_1Y_1+X_2Y_2+X_3Y_3$, Y_1^2, Y_2^2, Y_3^2, Y_4^2, Y_1Y_4, Y_2Y_4, Y_3Y_4, $Y_1Y_2+X_3Y_4$, $Y_2Y_3+X_1Y_4$, $Y_1Y_3+X_2Y_4$, *as k-algebra.*

Proof. By$[G_2]$ we have

$$4=e(R)\geqq 1+\sum_{i=1}^{d-1}\binom{d-1}{i-1}h^i(R),\ d=\dim R.$$

Since R is not Cohen-Macaulay $h_i(R)\neq 0$ for some $i\geqq 2$. Then we get $d\leqq 4$. Suppose that $d=4$.By [S], Satz (3.2.3), R is Cohen-Macaulay if and only if $h^2(R)=0$, because R is quasi-Gorenstein. Hence we have depth $R=2$ and $h^3(R)\neq 0$. But this yields $3\geqq 3h^2(R)+3h^3(R)$, which is impossible. Hence we have $d=3$. Let x_1, x_2, x_3 be a homogeneous system of parameters of degree 1 of R and let

$$J=(x_1, x_2) : x_3+(x_2,x_3) : x_1+(x_1,x_3) : x_2.$$

Then by $[G_2]$ we have

$$e(R)=4=1+l_R(M/J)+2h^2(R).$$

Therefore we get $h^2(R)=1$ and $I_R(M/J)=1$. Since R is Buchsbaum we have $(x_1, x_2):x_3/(x_1, x_2)\tilde{\to}H_M{}^2(R)(-2)$, cf. [G₂]. Since $h^2(R)=1$ we see that $(x_1, x_2):x_3=(x_1, x_2, y_3)$ for some homogeneous element y_3. If $n=\deg y_3\geqq 2$ we have $(x_2, x_3):x_1=(x_2, x_3, y_1)$ and $(x_1, x_3):x_2=(x_1, x_3, y_2)$ for some homogeneous elements y_1, y_2 with $\deg y_1=\deg y_2=n$. By the condition $l_R(M/J)=1$ we see that $v(R)\leqq 4$. But this implies that R is a hyper-surface, which contradicts to the assumption that R is not Cohen-Macaulay. Hence $\deg y_1=1$ and $H_M{}^2(R)=k(1)$. Therefore $M=(x_1, x_2, x_3, y_1, y_2, y_3, y)$ for some homogeneous element y_i of degree 1. Since J is generated by elements of degree 1 and since $l_R(M/J)=1$ we have $M^2=JM+yM=(x_1, x_2, x_3)M$. Therefore $v(R)=e(R)+d+I(R)-1=7$. Let $S=k[x_1, x_2, x_3]$. We are going to show that $K_R=R(-2)$. Since R is generated by 1, y_1, y_2, y_3, y as S-module we have a finite free resolution of R

$$(\#)\qquad 0\to S(-n)\overset{f}{\to}S\oplus S^4(-1)\to R\to 0.$$

Applying the functor $Hom_S(\ ,S(-3))$ to (#), we get the exact sequence

$$0\to K_R\to S(-3)\oplus S^4(-2)\overset{f^*}{\to}S(n-3)\to k(-1)\to 0,$$

by the local duality, cf. [GW]. This exact sequence shows that, after a change of basis, f can be written in the form $[0, 0, x_1, x_2, x_3]$ and that $n=2$.

Hence R is isomorphic to $F\oplus Z$ as S-module, where F is a free S-module of rank 2 and $\mathrm{rank}_S Z=2$. Let Z_2 be the second syzygy of k as graded S-module,i.e. $0\to Z_2\to S^3(-1)\to S\to k\to 0$ is exact. Then Z is isomorphic to $Z_2(1)$. it is easy to see that $F=S\oplus S(-1)$ and

$K_R\tilde{\to}Hom_S(F\oplus Z_2(1), S(-3))\tilde{\to}S(-3)\oplus S(-2)\oplus Z_2(-2)$, since Z_2 is isomorphic to $Hom_S(Z_2,S(-3))$. This show that K_R is generated by an element of degree 2. Hence $K_R=R(-2)$, as required. We are going to show that R has only one associated prime. To prove this we need the following lemma.

Lemma 11. *Let R be the same as in Theorem* 10. *Then we have*:

(1) *There is no ideal I such that $ht(I)=0$, $e(R/I)=2$ and R/I is Cohen-Macaulay.*

(2) *There is no radical ideal I such that* $ht(I)=0$, $e(R/I)=3$.

proof. *(1)* : Suppose that there is an ideal I such that $ht(I)=0$, $e(R/I)=2$ and R/I is Cohen-Macaulay. Let S be as in the proof Theorem 10. Then I is a reflexive S-module of rank 2 and $H_M{}^2(I)=H_M{}^2(R)=k(1)$, since R/I is Cohen-Macaulay. Hence

I is isomorphic to $Z_2(1)$ as S-module. From the exact sequence

$$0 \to I \to R \to R/I \to 0$$

we get an exact sequence,

$$0 \to K_{R/I} \xrightarrow{\phi} K_R \xrightarrow{\psi} Z_2(-1) \to 0.$$

Since $K_R = R(-2)$ the generator of K_R is mapped to a non-zero element of degree 2 by ψ, but there is no non-zero element of degree 2 in $Z_2(-1)$. This is absurd.

(2): Suppose that there is a radical ideal I such that $\mathrm{ht}(I)=0$ and $\mathrm{e}(R/I)=3$. From the exact sequence

$$0 \to I \to R \to R/I \to 0$$

we get an exact sequence

$$(\#\#) \qquad 0 \to K_{R/I} \to K_R \to K_I \to Ext_S^1(R/I, S(-3)) \to K(-1),$$

where $K_I = Hom_S(I, S(-3))$. Since $\mathrm{e}(R/I)=3$, I is a reflexive S-module of rank 1 and hence is free as S-module. Therefore $I \xrightarrow{\sim} R/p(-n)$ for some integer n and R/p is isomorphic to a polynomial ring in 3 variables over k. From (##) we have an exact sequence

$$R(-2) \xrightarrow{f} R/p(n-3) \to Ext_s^1(R/I, S(-3))$$

Suppose that $n \geqq 2$. Then $\mathrm{Coker}(f) \xrightarrow{\sim} R/(p,a)(n-3)$ for some $a \in R_{n-1}$.

Let E be the S_2-fication of R/I and let

$$0 \to R/I \to E \to C \to 0$$

be exact. From this exact sequence, we have an exact sequence

$$Ext_s^1(E, S(-3)) \to Ext_S^1(R/I, S(-3)) \to Ext_s^2(C, S(-3)).$$

Since E satisfies (S_2) the first term of this exact sequence is an R-module of finite length and the last term has dimension at most 1, because $\dim C \leqq 1$. Hence the dimension of $Ext_s^1(R/I, S(-3))$ is at most 1 but it contains $R/(p,a)(n-3)$ whose dimension is 2. This is absurd. Hence $n=1$. Now f must be onto. Hence $Ext_s^1(R/I, S(-3))$ is a submodule of $k(-1)$. The exact sequence $0 \to R/p(-1) \to R \to R/I \to 0$ tells us that depth $R/I=2$. Hence $Ext_S^1(R/I, S(-3)) \xrightarrow{\sim} k(-1)$ and therefore R/I is a Buchsbaum ring satisfying (S_2) and $\mathrm{e}(R/I)=3$. Therefore R/I is not reduced by Theorem 7, contradicting to the assumption that I is a radical ideal. This completes the proof of Lemma 11.

As a corollary of the proof we get:

Corollary 12. *Let* R *be the same as in Theorem* 10. *If* I *is an unmixed ideal with ht* $(I)=0$ and $e(R/I)=3$ *then* R/I is a Buchsbaum ring satifying (S_2).

Now let us continue the proof of Theorem 10.

By [N],(23.5) we have $e(R/p)\leqq 4$ for any associated prime p of R.

If $e(R/p)=4$ we see that R is a domain. By Lemma 5, we get

$$7=v(R)\leqq e(R)+\dim R-1=6,$$

a contradiction. By Lemma 11 there is no associated prime p such that $e(R/p)=2$ or 3. Hence $e(R/p)=1$ for any associated prime p of R. By Lemma 11,(2) the number of associated primes is at most 2. Suppose that $\mathrm{Ass}(R)=\{p_1, p_2\}$. Then, by Lemma 2 we get $\mathrm{ht}(p_1+p_2)=1$ and hence by Lemma 4 $R/p_1\cap p_2$ is Cohen-Macaulay. But this is impossible by Lemma 11,(1). Now we have shown that R has only one associated prime. Let p be the unique associated prime of R. Let y be a homogeneous element of R such that $p=(0:y)$ and let $n=\deg y$. We have an exact sequence

$$\begin{array}{l} 0\to yR\to R\to R/yR\to 0 \\ \quad\ \ \downarrow\wr \\ R/p(-n) \end{array}$$

,which yields that $e(R/yR)=3$ and that yR is an unmixed ideal of R. By Corollary 11 and Theorem 7 R/yR is isomorphic to

$$R'=k[X_1, X_2, X_3, Y_1, Y_2, Y_3]/(X_1Y_1+X_2Y_2+X_3Y_3, (Y_1, Y_2, Y_3)^2).$$

This shows that $\deg y=1$ and $p=(y_1, y_2, y_3, y)$, where y_i is an inverse image of the image of Y_i in R', and $R=k[x_1, x_2, x_3, y_2, y_2, y_3, y]$ for some $x_i\epsilon R$ which is mapped to the image of X_i in R'.

Let $X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4$ be indeterminates over k and let

$\phi: k[X_1, X_2, X_3, Y_1, Y_2, Y_3, Y_4]\to$R be a homomorphism given by $\phi(X_i)=x_i$, $\phi(Y_i)=y_i$ for $i=1,2,3$ and $\phi(Y_4)=y$.

Then $\mathrm{Ker}(\phi)$ is minimally generated by 11 elements of degree 2, by $[G_1]$, Theorem (1.1) and Theorem (4.4). Since $yp=(0)$ we get 4 relations

$$(1)\qquad y_1y=y_2y=y_3y=y^2=0.$$

And by the choice of x_i, y_i we get 7 relations

$$(2)\qquad x_1y_1+x_2y_2+x_3y_3+ay=0,$$

$$(3)\qquad y_iy_j-f_{ij}y=0 \qquad (1\leqq i\leqq j\leqq 3),$$

for some homogeneous element a, f_{ij} of degree 1.

By (1) we may assume that $a, f_{ij}\epsilon S=k[x_1, x_2, x_3]$ and (2) can be written in the form

(2)' $x_1y_1+x_2y_2+x_3y_3=0$,

after a suitable change of variables among y_i, y. Then we have a minimal free resolution of R as S-module

$$0\to S(-2)\xrightarrow{[0,\ x_1,\ x_2,\ x_3,\ 0,]} S\oplus S^4(-1)\xrightarrow{\psi} R\to 0,$$

where $\psi(e_0)=1$, $\psi(e_i)=y_i$ $(1\leqq i\leqq 4)$ for some free basis $e_0, e_1,\cdots,e_4$ of $S\oplus S^4(-1)$ with deg $e_0=0$, deg $e_i=1$ $(1\leqq i\leqq 4)$. Then we see that $K_R=Hom_S(R,S(-3))$, regarded as a submodule of $Hom_S(S\oplus S^4(-1),S(-3))$,is generated by $e_0{}^*$, $x_3e_2{}^*-x_2e_3{}^*$, $x_1e_3{}^*-x_3e_1{}^*$, $x_2e_1{}^*-x_1e_2{}^*$, $e_4{}^*$ as S-module, where $e_i{}^*$ is the dual base of e_i. Since $K_R=R(-2)$ and deg $e_0{}^*=3$, deg $e_i{}^*=2(1\leqq i\leqq 4)$ we conclude that K_R is generated by $e_4{}^*$ as R-module. Hence we have relations in K_R as R-module,

(4) $e_0{}^*=g_0e_4{}^*$

(5) $x_3e_2{}^*-x_2e_3{}^*=g_1e_4{}^*$

(6) $x_1e_3{}^*-x_3e_1{}^*=g_2e_4{}^*$

(7) $x_2e_1{}^*-x_1e_2{}^*=g_3e_4{}^*$

for some $g_i\in R$ with deg $g_i=1$.

Let

$$g_i=\sum_{i=1}^{3}\alpha_{ij}x_j+\sum_{j=1}^{4}\beta_{ij}y_i$$

for some α_{ij}, $\beta_{ij}\in k$. From (5)−(7) we get $x_1g_1+x_2g_2+x_3g_3=0$. From this we get

$$\sum_{1\leqq i\leqq j\leqq 3}(\alpha_{ij}+\alpha_{ji})x_ix_j+\sum_{i=1}^{3}\alpha_{ii}x_i^2+\sum_{j=1}^{4}(\sum_{i=1}^{3}\beta_{ij}x_i)y_j=0,$$

where $y_4=y$. Then we get

$\alpha_{ii}=0$ for $i=1,2,3$,

$\alpha_{ij}+\alpha_{ji}=0$ for $1\leqq i<j\leqq 3$,

$\beta_{ii}=\gamma$ for $i=1,2,3$,

for some $\gamma\in k$ and

$\beta_{ij}=0$ for $i\neq j$.

Therefore

$g_1=\alpha_{12}x_2+\alpha_{13}x_3+\gamma y_1$

$g_2=-\alpha_{12}x_1+\alpha_{23}x_3+\gamma y_2$

$g_3=-\alpha_{13}x_1-\alpha_{23}x_2+\gamma y_3.$

From (5) we get

$$
\begin{aligned}
0 &= (x_3 e_2{}^* - x_2 e_3{}^*)(y_4) \\
&= (g_4 e_4{}^*)(y_4) \\
&= e_4{}^*((\alpha_{12} x_2 + \alpha_{13} x_3) y_4) \\
&= \alpha_{12} x_2 + \alpha_{13} x_3,
\end{aligned}
$$

hence $\alpha_{12} = \alpha_{13} = 0$. Similarly we get $\alpha_{23} = 0$. Thus we have

$g_i = \gamma y_i$ for $i = 1,2,3$. In particular, $\gamma \neq 0$.

By evaluating the relations (4)−(7) at y_1, y_2, y_3, y_4, we get

$$
\begin{aligned}
&\gamma f_{ii} = 0 \text{ for } i = 1,2,3, \\
&x_3 = \gamma f_{12} = -\gamma f_{12}, \\
&x_2 = \gamma f_{13} = -\gamma f_{13}, \\
&x_1 = \gamma f_{23} = -\gamma f_{23}.
\end{aligned}
$$

This shows that $\mathrm{ch}(k) = 2$ and replacing $\gamma^{-1} y_4$ by y_4, we get the required relations. Q.E.D.

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