SOME PROPERTIES OF TORUS EQUIVARIANT VECTOR BUNDLES

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INTRODUCTION.

Let $E(a_0, \ldots, a_n)$ be a vector bundle of rank n on an n-dimensional projective space P^n defined by the exact sequence

$$0 \longrightarrow O_{\mathrm{P}^n} \longrightarrow \bigoplus_{i=0}^{n} O_{\mathrm{P}^n}(a_i) \longrightarrow E(a_0, \ldots, a_n) \longrightarrow 0,$$

where $a_0, \ldots, a_n$ are positive integers (See Definition 1). Let T be an n-dimensional algebraic torus. P^n has a natural action of T and becomes a toric variety. We consider the vector bundle on P^n with an equivariant torus action. It is called a torus-equivariant vector bundle. We find a detail definition of a torus-equivariant vector bundle in [1]. We know that an indecomposable torus-equivariant vector bundle of rank n on an n-dimensional projective space P^n is isomorphic to $E(a_0, \ldots, a_n)(l)$or $E(a_0, \ldots, a_n)^*(l)$ for some positive integers $a_0, \ldots, a_n$ and some integer l. Hence we consider the vector bundles $E(a_0, \ldots, a_n)\ (l)$ and $E(a_0, \ldots, a_n)^*\ (l)$.

§1 Definitins and notations.

DEFINITION 1. Let $a_0, \ldots, a_n$ $(n \geq 2)$ be positive integers. We define a vector bundle $E(a_0, \ldots, a_n)$ of rank n on n-dimensional projective space P^n by the exact sequence

$$0 \longrightarrow O_{\mathrm{P}^n} \xrightarrow{f} \bigoplus_{i=0}^{n} O_{\mathrm{P}^n}(a_i) \longrightarrow E(a_0, \ldots, a_n) \longrightarrow 0$$

such that f sends 1 to $(X_0^{a_0}, X_1^{a_1}, \ldots, X_n^{a_n})$ where $X_0, \ldots, X_n$ are homogeneous coordinates of P^n.

DEFINITION 2. We define a numerical function $g(b)$ for an integer b by

$$g(b) = \begin{cases} \frac{1}{n!}(b+1)(b+2)\ldots(b+n) & \text{if } b \geq 0. \\ 0 & \text{if } b < 0. \end{cases}$$

LEMMA 3. *Let* $E = E(a_0, \ldots, a_n)$ *be a vector bundle of rank* n *on* P^n. *Then, for an integer* l, *we have*

(1) $\dim H^0(P^n, E^*(l)) = \sum_{i<j} g(l-a_i-a_j) - \sum_{i<j<h} g(l-a_i-a_j-a_h)$

$$+\sum_{i<j<h<k} g(l-a_i-a_j-a_h-a_k) - \ldots$$

(2) $\dim H^0(P^n, E(l)) = \sum_{i=0}^{n} g(l+a_i) - g(l)$

This lemma is proved in [2].

§2 Isomorphism of torus equivariant vector bundles.

THEOREM 4. *Let* $a_0, \ldots, a_n$ *and* $b_0, \ldots, b_n$ *be positive integers such that* $0<a_0\leq a_1\leq \ldots \leq a_n$, $0<b_0\leq b_1\leq \ldots \leq b_n$ $(n\geq 2)$.
Assume $E(a_0, \ldots, a_n)(l)\simeq E(b_0, \ldots, b_n)(m)$ *then we have*

$$a_0=b_0,\ a_1=b_1,\ \ldots,\ a_n=b_n \text{ and } l=m.$$

Proof. Put $E=E(a_0, \ldots, a_n)$. $F=E(b_0, \ldots, b_n)$. We first prove the case of $l = m=0$. Assume $E \simeq F$. Then we have

$$\dim H^0(P^n, E(-k))=\dim H^0(P^n, F(-k))$$

for a positive integer k. Since, by Lemma 3,

$$\begin{aligned} \dim H^0(P^n, E(-k)) &= \sum_{i=0}^{n} g(a_i-k) \\ \dim H^0(P^n, F(-k)) &= \sum_{i=0}^{n} g(b_i-k) \end{aligned} \qquad (*)$$

we have

$$\sum_{i=0}^{n} g(a_i-k) = \sum_{i=0}^{n} g(b_i-k) \qquad (**)$$

Now we see that $a_n=b_n$. We take $k=b_n+1$ in the equations $(*)$ when $a_n>b_n$ then

$$\dim H^0(P^n, E(-k))\neq 0,\quad \dim H^0(P^n, F(-k))=0,$$

a contradiction, while we take $k=a_n+1$ in the equations $(*)$ when $a_n<b_n$ then

$$\dim H^0(P^n, E(-k))=0,\quad \dim H^0(P^n, F(-k))\neq 0,$$

a contradiction. Hence we have $a_n=b_n$. So we prove $a_i=b_i$ $(i=n, \ldots, 0)$ by the descending induction on i. Assume that $a_i=b_i$ for $i\geq t+1$. We take $k=b_t+1$ in the equation $(**)$ when $a_t>b_t$ then

$$\sum_{i=0}^{n} g(a_i-b_t-1) = \sum_{i=0}^{n} g(b_i-b_t-1)$$

hence, by the inductive hypothesis, we have

$$\sum_{i=0}^{t} g(a_i - b_t - 1) = \sum_{i=0}^{t} g(b_i - b_t - 1).$$

This is a contradiction since the left hand side is non-zero and the right hand side is zero. While we take $k = a_t + 1$ in the equation $(**)$ when $a_t < b_t$ then

$$\sum_{i=0}^{n} g(a_i - a_t - 1) = \sum_{i=0}^{n} g(b_i - a_t - 1),$$

hence, by the inductive hypothesis, we have

$$\sum_{i=0}^{t} g(a_i - a_t - 1) = \sum_{i=0}^{t} g(b_i - a_t - 1).$$

This is a contradiction by the same reason. Therefore we have $a_t = b_t$. Hence, by the induction, we have $a_i = b_i$ for all i.

We now prove the case of $E(l) \simeq F(m)$. Since $E \simeq F(m-l)$ or $E(l-m) \simeq F$ we may assume that $E \simeq F(m)$ for $m \geq 0$. Then we have

$$\dim H^0(P^n, E(-k)) = \dim H^0(P^n, F(m-k))$$

for a positive integer k. Since

$$\begin{aligned} &\dim H^0(P^n, E(-k)) = \sum_{i=0}^{n} g(a_i - k) \\ &\dim H^0(P^n, F(m-k)) = \sum_{i=0}^{n} g(b_i + m - k) \end{aligned} \qquad (***)$$

we have

$$\sum_{i=0}^{n} g(a_i - k) = \sum_{i=0}^{n} g(b_i + m - k) \qquad (****)$$

We first see that $a_n = b_n + m$. We take $k = b_n + m + 1$ in the equation $(***)$ when $a_n > b_n + m$ then we have

$$\dim H^0(P^n, E(-k)) \neq 0, \quad \dim H^0(P^n, F(m-k)) = 0,$$

a contradiction. While we take $k = a_n + 1$ in the equations $(***)$ when $a_n < b_n + m$ then we have

$$\dim H^0(P^n, E(-k)) = 0, \dim H^0(P^n, F(m-k)) \neq 0,$$

a contradiction. Hence we have $a_n = b_n + m$. We prove $a_i = b_i + m$ by the descending induction on i. Assume that $a_i = b_i + m$ for $i \geq t+1$. We take $k = b_t + m + 1$ in the equation $(****)$ when $a_t > b_t + m$ then we have

$$\sum_{i=0}^{n} g(a_i - b_t - m - 1) = \sum_{i=0}^{n} g(b_i - b_t - 1).$$

Hence, by the inductive hypothesis, we have

$$\sum_{i=0}^{t} g(a_i - b_t - m - 1) = \sum_{i=0}^{t} g(b_i - b_t - 1).$$

This is a contradiction since the left hand side is non-zero and the right hand side is zero. While we take $k = a_t + 1$ in the equation $(****)$ when $a_t < b_t + m$ then we have

$$\sum_{i=0}^{n} g(a_i - a_t - 1) = \sum_{i=0}^{n} g(b_i + m - a_t - 1).$$

Hence, by the inductive hypothesis, we have

$$\sum_{i=0}^{t} g(a_i - a_t - 1) = \sum_{i=0}^{t} g(b_i + m - a_t - 1).$$

This is a contradiction. Therefore we have $a_t = b_t + m$. Hence, by the induction, we have $a_i = b_i + m$ for all i. Then we have an equation

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b_i + (n+1)m.$$

Since deg $E =$ deg $F(m)$ we have

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b_i + nm.$$

Hence $m = 0$ and $a_i = b_i$ for all i Q. E. D.

§3 Isomorphism of dual torus equivariant vector bundles.

In this section, we consider the case such that $E \simeq F^*(l)$ where $E = E(a_0 \ldots\ldots a_n)$ $(0 \leq a_0 \leq a_1 \leq \ldots \leq a_n)$ and $F = E(b_0 \ldots\ldots \mathrm{b}_n)(0 \leq b_0 \leq b_1 \leq \ldots \leq b_n)(n \geq 2)$. We note that $E \simeq F^*(l)$ means $F \simeq E^*\ (l)$ by taking the dual.

LEMMA 5. *Assume* $E \simeq F^*(l)$. *Then* $nl = \sum_{i=0}^{n} a_i + \sum_{i=0}^{n} b_i$.

Proof. Since deg $E = \sum_{i=0}^{n} a_i$ and deg $F^*(l) = nl - \sum_{i=0}^{n} b_i$ we have the conclusion.

Q. E. D.

LEMMA 6. *Assume* $E \simeq F^*(l)$. *Then* $l = a_n + b_1 + b_0$.

Proof. Since

$$\dim \mathrm{H}^0(\mathrm{P}^n, E(-k)) = \dim \mathrm{H}^0(\mathrm{P}^n, F^*(l-k),$$

for a positive integer k and

$$\dim \mathrm{H}^0(\mathrm{P}^n, E(-k)) = \sum_{i=0}^{n} g(a_i - k),$$

$$\dim \mathrm{H}^0(\mathrm{P}^n, F(l-k)) = \sum_{i<j} g(l-k-b_i-b_j) - \sum_{i<j<h} g(l-k-b_i-b_j-b_h) + \ldots$$

we have that $l-k-b_0-b_1<0$ implies $a_n-k<0$. Hence we have $a_n \leq l-b_0-b_1$. Furthermore $l-k-b_0-b_1 \geq 0$ and $l-k-b_0-b_1-b_2<0$ implies $a_n-k \geq 0$. Hence we have $a_n \geq l-b_0-b_1$. Therefore we have $a_n=l-b_0-b_1$ i. e. $l=a_n+b_1+b_0$. Q. E. D.

LEMMA 6'. *Assume* $E \simeq F^*(l)$. *Then* $l=b_n+a_1+a_0$.

LEMMA 7. *Assume* $E \simeq F^*(l)$. *If* $a_n-a_{n-1}>p$ *then* $b_2-b_1>p$.

Proof. If $p<0$ then the assertion is clear because $a_n \geq a_{n-1}$ and $b_2 \geq b_1$. We prove by induction on p. Let $p \geq 0$. Since

$$\dim H^0(P^n, E(p-a_n)) = \dim H^0(P^n, F^*(l+p-a_n)) = \dim H^0(P^n, F^*(p+b_0+b_1))$$

and $p+b_0+b_1-b_i-b_j-b_h<0$ $(0 \leq i \leq j \leq h)$ by the inductive hypothesis, we have that

$$\sum_{i=0}^{n} g(p-a_n+a_i)-g(p-a_n)$$
$$= \sum_{i<j} g(p+b_0+b_1+b_i-b_j) - \sum_{i<j<h} g(p+b_0+b_1-b_i-b_j-b_h) + \ldots$$

i. e. $\quad g(p)=\sum_{i<j} g(p+b_0+b_1-b_i-b_j)$.

Hence we have $g(p+b_0+b_1-b_0-b_2)=0$. This means that $p+b_0+b_1-b_0-b_2<0$ i. e. $b_2-b_1>p$. Q. E. D.

LEMMA 7'. *Assume* $E \simeq F^*(l)$. *If* $b_n-b_{n-1}>p$ *then* $a_2-a_1>p$.

LEMMA 8. *Assume* $E \simeq F^*(l)$. *If* $a_n-a_{n-1}=p$ *then* $b_2-b_1=p$.

Proof. Since $a_n-a_{n-1}>p-1$ we have $b_2-b_1 > p-1$ by Lemma 7. Hence we have that

$$p+b_0+b_1-b_i-b_j-b_h<0$$

for $0 \leq i<j<h$. Since

$$\dim H^0(P^n, E(p-a_n)) = \dim H^0(P^n, F^*(l+p-a_n)) = \dim H^0(P^n, F^*(p+b_0+b_1))$$

we have

$$mg(0)+g(p) = \sum_{i<j} g(p+b_0+b_1-b_i-b_j) - \sum_{i<j<h} g(p+b_0+b_1-b_i-b_j-b_h) + \ldots$$
$$= \sum_{i<j} g(p+b_0+b_1-b_i-b_j)$$

where $m=\# \{ i \mid a_i=a_{n-1} (0 \leq i \leq n) \}$. Hence this means that

$$p+b_0+b_1-b_0-b_2 \geq 0 \quad \text{i. e.} \quad b_2-b_1 \leq p.$$

Therefore we have $b_2 - b_1 = p$ since $b_2 - b_1 > p - 1$ by Lemma 7. Q. E. D.

LEMMA 8'. *Assume* $E \simeq F^*(l)$. *If* $b_n - b_{n-1} = p$ *then* $a_2 - a_1 = p$.

THEOREM 9. *Assume* $E \simeq F^*(l)$ *and* $n = 2$. *Then*

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2 \text{ and } l = a_0 + a_1 + a_2.$$

Proof. By Lemma 6 and Lemma 6' the following equations hold.

$$l = a_2 + b_1 + b_0 \tag{1}$$

$$l = b_2 + a_1 + a_0 \tag{2}$$

By Lemma 8 and Lemma 8' we can put

$$a_2 = a_1 + p,\ b_2 + b_1 = p \tag{3}$$

From (1), (2) and (3) we have $a_0 = b_0$. We now prove $a_1 = b_1$. Assume $a_1 < b_1$. Since

$$\dim H^0(P^2, E) = \dim H^0(P^2, F^*(l))$$

we have

$$\sum_{i=0}^{n} g(a_i) - 1 = \sum_{i<j} g(l - b_i - b_j) - g(l - b_0 - b_1 - b_2).$$

Since, by Lemma 6,

$$l - b_0 - b_1 = a_2,\ l - b_0 - b_2 = a_1,$$

$$l - b_1 - b_2 = a_0 + a_1 - b_1,\ \ l - b_0 - b_1 - b_2 = a_1 - b_1$$

we have

$$g(a_0) - 1 = g(a_0 + a_1 - b_1).$$

This is a contradiction since $a_0 > a_0 + a_1 - b_1$ and the definition of the function g.

Assume $a_1 > b_1$. We consider $F \simeq E^*(l)$. Then we have a cotradiction by the same way. Hence we have $a_1 = b_1$. Therefore $a_2 = b_2$ and $l = a_0 + a_1 + a_2$. Q. E. D.

Remark. Let E be a rank two vector bundle on P^2. Then $E \simeq E^*(l)$ ($l = \deg E$). Hence Theorem 9 is satisfied.

THEOREM 10. *When* $n = 3$, 4 *there exists no isomorphism* $E \simeq F^*(l)$.

Proof. Assume $E \simeq F^*(l)$ and $n = 3$. Then by Lemma 5, Lemma 6 and Lemma 6' we have

$$3l = a_0 + a_1 + a_2 + a_3 + b_0 + b_1 + b_2 + b_3,$$

$$l = a_3 + b_0 + b_1 = b_3 + a_0 + a_1.$$

Hence These mean $l = a_2 + b_2$. By Lemma 8 and Lemma 8', we can put

$$a_3 = a_2 + p,\ b_2 = b_1 + p$$

and

$$a_2 = a_1 + q,\ \ b_3 = b_2 + q.$$

Then

$$\begin{aligned} 3l &= (a_0 + a_1 + a_2 + a_3) + (b_0 + b_1 + b_2 + b_3) \\ &= a_0 + a_1 + (a_1 + q) + (a_1 + p + q) + b_0 + b_1 + (b_1 + p) + (b_1 + p + q) \\ &= a_0 + b_0 + 3\,(a_1 + b_1 + p + q) \\ &= a_0 + b_0 + 3(a_2 + b_2) \\ &= a_0 + b_0 + 3l. \end{aligned}$$

This means $a_0 + b_0 = 0$, a contradiction for $a_0 > 0$ and $b_0 > 0$.

Assume $E \simeq F^*(l)$ and $n = 4$. By Lemma 5, Lemma 6 and Lemma 6' we have

$$4l = (a_0 + a_1 + a_2 + a_3 + a_4) + (b_0 + b_1 + b_2 + b_3 + b_4)$$

$$l = a_4 + b_0 + b_1 = b_4 + a_0 + a_1.$$

By Lemma 8 and Lemma 8' we can put

$$a_4 = a_3 + p,\ \ b_2 = b_1 + p$$

and

$$a_2 = a_1 + q,\ \ b_4 = b_3 + q.$$

Then, since

$$l = a_4 + b_0 + b_1 = a_3 + p + b_0 + b_1 = a_3 + b_2 + b_0,$$

$$l = b_4 + a_0 + a_1 = b_3 + q + a_0 + a_1 = b_3 + a_2 + a_0$$

we have

$$\begin{aligned} 4l &= l + l + l + l \\ &= (a_4 + b_0 + b_1) + (a_3 + b_2 + b_0) + (b_4 + a_0 + a_1) + (b_3 + a_2 + a_0) \\ &= (a_0 + a_1 + a_2 + a_3 + a_4 + b_0 + b_1 + b_2 + b_3 + b_4) + (a_0 + b_0) \\ &= 4l + (a_0 + b_0). \end{aligned}$$

This means $a_0 + b_0 = 0$, a contradiction for $a_0 > 0$ and $b_0 > 0$. Q. E. D.

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