Torus-Equivariant Vector Bundles of Rank Three on $\mathbf{P}^2$ (I)

Tamafumi Kaneyama

(Received April 30, 1987)

Introduction.

Let T be a two dimensional algebraic torus over an algebraically closed field K. Then $\mathbf{P}^2$ has a natural action of T and becomes a toric variety. Let E be a vector bundle on $\mathbf{P}^2$. We say that E is T-equivariant if, for every K-rational point t of T, there exists an isomorphism $f_t : t^*E \longrightarrow E$ where $t : \mathbf{P}^2 \longrightarrow \mathbf{P}^2$ is the action of t on $\mathbf{P}^2$. A T-equivariant vector bundle has a T-liniarization (see Definition 1. 2 and [1]). We consider a T-equivariant vector bundle on $\mathbf{P}^2$. We know indecomposable T-equivariant vector bundles of rank two on $\mathbf{P}^2$ in [1]. In this paper, we generalize this method and classify indecomposable T-equivariant vector bundles of rank three on $\mathbf{P}^2$.

§1. Preliminaries of *T*-equivariant vector bundles.

Let N be a free $\mathbf{Z}$-module of rank n and let M be the dual $\mathbf{Z}$-module of N. Then there is a natural $\mathbf{Z}$-bilinear map

$$\langle\ ,\ \rangle : M \times N \longrightarrow \mathbf{Z}.$$

It can naturally be extended to $M_{\mathbf{R}} \times N_{\mathbf{R}} \longrightarrow \mathbf{R}$ where $M_{\mathbf{R}} = M \otimes_{\mathbf{Z}} \mathbf{R}$ and $N_{\mathbf{R}} = N \otimes_{\mathbf{Z}} \mathbf{R}$. We denote $\varphi(\xi) = \langle \xi, \varphi \rangle$ for ξ in $M_{\mathbf{R}}$ and φ in $N_{\mathbf{R}}$. Let $T = T_N$ be an n-dimensional algebraic torus over an algebraically closed field K defined by N i.e.,

$$T_N = N \otimes_{\mathbf{Z}} K^* = Hom_{\mathbf{Z}}(M, K^*)$$

where K^* is the multiplicative group of K. Then we can identify M with the additive group of characters of T. Let $X = T_N\mathrm{emb}(\Delta)$ be a smooth complete toric variety of dimension n defined by a fan Δ of N for which the reader is referred to [3, 4].

DEFINITION 1.1. An equivariant vector bundle E on X is a vector bundle on X such that there exists an isomorphism $f_t : t^*E \longrightarrow E$ for every K-rational point t in T. where $t : X \longrightarrow X$ is the action of t on X.

DEFINITION 1.2. An equivariant vector bundle $E = (E, f_t)$ is said to be T-linearized if

$f_{tt'} = f_t \circ t'^*(f_t)$ holds for every pair of K-rational points t, t' of T, where

$$f_{tt'} = f_t \circ t'^*(f_t) : (tt')^*E \xrightarrow{t'^*(f_t)} t'^*E \xrightarrow{f_{t'}} E.$$

In [1], we showed that an equivariant vector bundle necessarily has a T-linearization. We also studied how to describe T-linearized vector bundle in terms of fans, as we now recall. Let Δ be a fan of N and we denote by $\Delta(l)$ the set of an l-dimensional cones. For C in $\Delta(l)$, there exists a finite subset $\{\varphi_1, \ldots, \varphi_l\}$ of N and $C = \mathbf{R}_0\varphi_1 + \ldots + \mathbf{R}_0\varphi_l$ where $\mathbf{R}_0$ is the set of non-nagative real numbers. We say that $\{\varphi_1, \ldots, \varphi_l\}$ is the fundamental system of generators of C if φ_i are primitive i.e., φ_i is not a non-trivial integral multiple of any element of N. The fundamental system of generators $\{\varphi_1, \ldots, \varphi_l\}$ of C is uniquely determined by C and is denoted by $|C|$. We consider the following :

(I) $m : \{|C| \mid C \in \Delta(1)\} \longrightarrow \mathbf{Z}^{\oplus r}$

sending φ to $m(\varphi) = \{m(\varphi)_1, \ldots, m(\varphi)_r\}$, and for every C in $\Delta(n)$,

$m_c : |C| \longrightarrow \mathbf{Z}^{\oplus r}$

so that there exists a permutation $\tau = \tau_c$ such that

$m_c(\varphi) = (m_c(\varphi)_1, \ldots, m_c(\varphi)_r) = (m(\varphi)_{\tau(1)}, \ldots, m(\varphi)_{\tau(r)}\}$

for every φ in $|C|$.

Let C be an n-dimensional cone in $\Delta(n)$. Then we have a set of characters $\{\xi(C)_1, \ldots, \xi(C)_r\}$ in M by solving, for each $1 \leq i \leq r$, the equations $\varphi(\xi(C)_i) = m_c(\varphi)_i$ for every φ in $|C|$. Then it is easy to see that (I) is equivalent to the following :

(I') $\xi : \Delta(n) \longrightarrow M^{\oplus r}$

sending C to $\xi(C) = (\xi(C)_1, \ldots, \xi(C)_r)$ such that there exists a permutation $\tau = \tau_{c,c'}$ for every pair of cones C and C' in $\Delta(n)$, so that $\varphi(\xi(C)_i) = \varphi(\xi(C')_{\tau(i)})$ for every i and every φ in $|C| \cap |C'|$.

(II) $P : \Delta(n) \times \Delta(n) \longrightarrow GL_r(K)$

sending (C, C') to $P(C, C') = (P(C,C')_{ij})$ such that $P(C, C')_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C')_j)$ for every φ in $|C| \cap |C'|$ and that

$P(C, C')P(C', C'') = P(C, C'')$

for every C, C', C'' in $\Delta(n)$.

For (m, P) defined by (I) and (II), we denote by $E(m, P)$ the T-linearized vector bundle obtained from (m, P). We refer the reader to [4] as for the construction of the T-linearized vector bundle $E(m, P)$.

(III) Two pairs $(m, P)$and (m', P') defined by (I) and (II) are said to be equivalent if there exists a permutation $\tau = \tau_c$ for every C in $\Delta(n)$ such that

$$(m_C(\varphi)_1, \ldots, m_C(\varphi)_r) = (m'_C(\varphi)_{\tau(1)}, \ldots, m'_C(\varphi)_{\tau(r)})$$

for every φ in $|C|$ and if there exists

$$\sigma : \Delta(n) \longrightarrow GL_r(K)$$

such that $\sigma(C)_{ij} \neq 0$ only if $\varphi(\xi(C)_i) \geq \varphi(\xi(C)_j)$ for every φ in $|C|$ and such that

$$P'(C, C') = \sigma(C)^{-1}P(C, C')\sigma(C')$$

holds for every C and C' in $\Delta(n)$.

THEOREM 1.3. Let $X = T_N\mathrm{emb}(\Delta)$ be a smooth complete toric variety defined by a fan Δ. Then the set of T-linearized vector bundles of rank r up T-isomorphism corresponds bijectively to the set of (I) (or (I ')) and (II) up to the equivalence (III).

REMARK 1.4. Let D_φ be the divisor corresponding to the cone $\mathbf{R}_0\varphi$ in $\Delta(1)$. Put $m_\varphi = m(\varphi)$ where m is defined by (I) in the case $r = 1$. Let $P(C, C') = 1$ for every C and C' in $\Delta(n)$. Then the T-linearized vector bundle $E(m,P)$ is the line bundle $O_X(-\Sigma m_\varphi D_\varphi)$, where the summantion is taken over φ in $\{|C| \mid C \in \Delta(1)\}$.

REMARK 1.5. Let $E = E(m,P)$ be the T-linearized vector bundle of rank r defined by (m,P). Then $E \otimes O_X(-\Sigma m_\varphi D_\varphi)$ is T-isomorphic to $E(m', P)$, where

$$m'(\varphi) = (m(\varphi)_1 + m_\varphi, \ldots, m(\varphi)_r + m_\varphi)$$

for every φ in $\{|C| \mid C \in \Delta(1)\}$. The dual vector bundle E^* is T-isomorphic to $E(-m, {}^tP^{-1})$, where

$${}^tP^{-1}(C, C') = {}^tP(C,C')^{-1}$$

and

$$-m(\varphi) = (-m(\varphi)_1, \ldots, -m(\varphi)_r)$$

for every φ in $\{|C| \mid C \in \Delta(1)\}$.

§2. *T*-equivariant vector bundles on $\mathbf{P}^2$.

Let N be a free $\mathbf{Z}$-module of rank two. Let $\{\varphi_1, \varphi_2\}$ be a $\mathbf{Z}$-base of N and put $\varphi_0 = -\varphi_1 - \varphi_2$ and

$$C = \mathbf{R}_0\varphi_1 + \mathbf{R}_0\varphi_2, \quad C' = \mathbf{R}_0\varphi_2 + \mathbf{R}_0\varphi_0, \quad C'' = \mathbf{R}_0\varphi_0 + \mathbf{R}_0\varphi_1.$$

Let Δ be a fan of N defined by $\{\varphi_0, \varphi_1, \varphi_2\}$ i. e.,

$$\Delta(2) = \{C, C', C''\}.$$

Then $\mathbf{P}^2 = T_N\mathrm{emb}(\Delta)$. From now on we restrict ourselves to the case $\mathbf{P}^2$.

LEMMA 2.1. Let $E = E(m,P)$ be an T-equivariant vector bundle of rank r on $\mathbf{P}^2$. By rearranging $\{\xi(C)_i\}$, $\{\xi(C')_i\}$ and $\{\xi(C'')_i\}$ and replacing (m, P) by an equivalent pair, we may assume that

(1) $P(C, C'')$ is an upper triangular matrix.

(2) $P(C, C'')_{ii} = 1$ for $1 \leq i \leq r$.

(3) $P(C', C)_{ii} = 1$ for $1 \leq i \leq r$.

(4) $\varphi_1(\xi(C)_1) \geq \varphi_1(\xi(C)_2) \geq \ldots \geq \varphi_1(\xi(C)_r)$.

(5) $\varphi_1(\xi(C)_i) = \varphi_1(\xi(C'')_i)$ for $1 \leq i \leq r$.

(6) $\varphi_2(\xi(C)_i) = \varphi_2(\xi(C')_i)$ for $1 \leq i \leq r$.

Proof. By rearranging $\{\xi(C)_i\}$ and $\{\xi(C'')_i\}$ we may assume $P(C, C'')_{ii} \neq 0$ for $1 \leq i \leq r$. Then, since $\varphi_1(\xi(C)_i) \geq \varphi_1(\xi(C'')_i)$ for $1 \leq i \leq r$, we have $\varphi_1(\xi(C)_i) = \varphi_1(\xi(C'')_i)$ for $1 \leq i \leq r$ by (I) in §1. Suppose $P(C, C'')_{hk} \neq 0$ and $P(C, C'')_{kh} = 0$ for some $h > k$, then, by interchanging $\xi(C)_h$ with $\xi(C)_k$ as well as $\xi(C'')_h$ with $\xi(C'')_k$, we have $P(C, C'')_{hk} = 0$ and $P(C, C'')_{kh} \neq 0$. Suppose $P(C, C'')_{hk} \neq 0$ and $P(C, C'')_{kh} \neq 0$ for some h, k ($h > k$) then we have $\varphi_1(\xi(C)_h) \geq \varphi_1(\xi(C'')_k)$ and $\varphi_1(\xi(C)_k) \geq \varphi_1(\xi(C'')_h)$, hence $\varphi_1(\xi(C)_h) = \varphi_1(\xi(C)_k)$. If $\varphi_2(\xi(C)_h) < \varphi_2(\xi(C)_k)$ then we interchange $\xi(C)_h$ with $\xi(C)_k$. So we have $\varphi_2(\xi(C)_h) \geq \varphi_2(\xi(C)_k)$. We define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = h,\ j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $P(C, C'')_{hk} = 0$. After finite times, we will reach an upper triangular matrix $P(C, C'')$. Hence (1), (4) and (5) are satisfied. Since $\det(P(C', C)) \neq 0$, by rearranging $\{\xi(C')_i\}$, we have $P(C', C)_{ii} \neq 0$ for every i. Hence (6) is satisfied.

Furthermore by replacing (m, P) by an equivalent pair, we may assume that $P(C, C'')_{ii} = 1$ and $P(C', C)_{ii} = 1$ for $1 \leq i \leq r$. Hence (2) and (3) are satisfied.

Q. E. D.

PROPOSITION 2.2. Let $E = E(m,P)$ be a T-equivariant vector bundle of rank r ($r > 1$) on $\mathbf{P}^2$. Suppose that there exists an integer h such that

$$\varphi_0(\xi(C')_h) = \varphi_0(\xi C'')_h),\ \varphi_1(\xi(C)_h) = \varphi_1(\xi(C'')_h),\ \varphi_2(\xi(C)_h) = \varphi_2(\xi(C'')_h)$$

and $\varphi_2(\xi(C)_h)$ is a maximal element in $\{\varphi_2(\xi(C)_1), \ldots, \varphi_2(\xi(C)_r)\}$.

Then we have $E = L \oplus E'$ where L is a T-linearized line bundle and E' is a T-linearized vector bundle of rank r-1.

Proof. By Lemma 2.1 we may assume $h = 1$. Put

$$P(C',C) = \begin{pmatrix} 1 & & \\ p_1 & & \\ \vdots & * & \\ p_r & & \end{pmatrix}$$

then

$$P(C,C'') = P(C', C)P(C,C'') = \begin{pmatrix} 1 & & \\ p_1 & & \\ \vdots & * & \\ p_r & & \end{pmatrix}$$

Assume $p_k \neq 0$ for some $k \geq 2$, then we have

$\varphi_2(\xi(C')_k) \geq \varphi_2(\xi(C)_1) = \varphi_2(\xi(C')_1)$,

$\varphi_0(\xi(C')_k) \geq \varphi_0(\xi(C'')_1) = \varphi_0(\xi(C')_1)$.

Then we define $\sigma(C') = (\sigma(C')_{ij})$ by

$$\sigma(C')_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = k,\ j = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m,P) by an equivalent pair using this $\sigma(C')$. Then we can replace ourselves to the case $p_k = 0$. Doing the same we have $p_2 = \ldots = p_r = 0$.

Now put

$$P(C', C) = \begin{pmatrix} 1 & q_2 \ldots q_r \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}$$

Assume $q_k \neq 0$ for some $k \geq 2$, then we have $\varphi_2(\xi(C')_1) \geq \varphi_2(\xi(C)_k)$ hence $\varphi_2(\xi(C)_1) \geq \varphi_2(\xi(C)_k)$. Since $\varphi_1(\xi(C)_1) \geq \varphi_1(\xi(C)_k)$, we define $\sigma(C) = (\sigma(C)_{ij})$ by

$$\sigma(C)_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = 1,\ j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C)$. Then we can reduce ourselves to the case $q_k = 0$. Hence we have $q_2 = \ldots = q_r = 0$. Put

$$P(C, C'') = \begin{pmatrix} 1 & d_2 & \cdot & \cdot & \cdot & d_r \\ & 1 & & & & \\ & & \cdot & & * & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & 1 \end{pmatrix}$$

then we have

$$P(C', C'') = P(C', C)P(C, C'') = \begin{pmatrix} 1 & d_2 & \dots & d_r \\ 0 & & & \\ \vdots & & * & \\ \vdots & & & \\ 0 & & & \end{pmatrix}.$$

Assume $d_k \neq 0$ for some $k \geq 2$, then we have

$\varphi_1(\xi(C)_1) \geq \varphi_1(\xi(C'')_k)$ and $\varphi_0(\xi(C')_1) \geq \varphi_0(\xi(C'')_k)$.

Hence we have

$\varphi_1(\xi(C'')_1) \geq \varphi_1(\xi(C'')_k)$ and $\varphi_0(\xi(C'')_1) \geq \varphi_0(\xi(C'')_k)$.

We define $\sigma(C'') = (\sigma(C'')_{ij})$ by

$$\sigma(C'')_{ij} = \begin{cases} 1 & \text{for } i = j, \\ c \neq 0 & \text{for } i = 1, j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and replace (m, P) by an equivalent pair using this $\sigma(C'')$. Then we can replace ourselves to the case $d_k \neq 0$. Hence $d_2 = \dots = d_r = 0$.

Thus we have that $P(C', C)$, $P(C, C'')$ and $P(C'', C')$ are of the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ \vdots & & & \\ 0 & & & \end{pmatrix}.$$

This means that $E = L \oplus E'$.

Q. E. D.

COROLLARY 2.3. Let $E = E(m, P)$ be a T-equivariant vector bundle of rank r $(r > 1)$ on $\mathbf{P}^2$. Suppose that there exist an integer h such that

$\varphi_0(\xi(C')_h) = \varphi_0(\xi(C'')_h)$, $\varphi_1(\xi(C)_h) = \varphi_1(\xi(C'')_h)$, $\varphi_2(\xi(C)_h) = \varphi_2(\xi(C'')_h)$

and $\varphi_2(\xi(C)_h)$ is a minimal element in $\{\varphi_2(\xi(C)_1), \dots, \varphi_2(\xi(C)_r)\}$.

Then we have $E = L \oplus E'$ where L is a T-inearized line bundle and E' is a T-linearized vector bundle of rank r-1.

Proof. We consider the dual T-equivariant vector bundle then we can apply Proposition 2.2.

Q. E. D.

From now on we consider a T-equivariant vector bundle of rank three. Put

$m_C(\xi_1) = m_{C''}(\varphi_1) = (a_1, a_2, a_3)$ $(a_1 \geq a_2 \geq a_3)$,

$m_C(\varphi_2) = m_{C'}(\varphi_2) = (b_1, b_2, b_3)$.

$m_{C'}(\varphi_0) = (c_1, c_2, c_3)$, $m_{C''}(\varphi_0) = (c'_1, c'_2, c'_3)$.

We note that the two sets $\{c_1, c_2, c_3\}$ and $\{c'_1, c'_2, c'_3\}$ are same sets.

LEMMA 2.4. Assume that two elements of $\{a_1, a_2, a_3\}$ are equal integers. Then E is decomposable.

Proof. Now we assume $a_1 = a_2$. If $c_1 = c'_1$ or $c_2 = c'_2$ then E is decomposable by Proposition 2.2. If $c_1 = c'_2$ or $c_2 = c'_1$ then we interchange $\xi(C'')_1$ with $\xi(C'')_2$. Thus we can apply Proposition 2.2. Hence E is decomposable. Assume $a_2 = a_3$ then we can apply Corollary 2.3.

Q. E. D.

By rearranging $\{\xi(C')_i\}$ (resp. $\{\xi(C'')_i\}$) we can apply Lemma 2.4 for φ_2 (resp. φ_0). i. e., two elements of $\{b_1, b_2, b_3\}$ (resp. $\{c_1, c_2, c_3\}$) are equal then E is decomposable. Therefore we assume that, for every i $(0 \leq i \leq 2)$, $\{m(\varphi_i)_1, m(\varphi_i)_2, m(\varphi_i)_3\}$ is the set of different integers.

LEMMA 2.5. Assume that $a_1 > a_2 > a_3$ and $b_1 > b_2 > b_3$. Then E is decomposable.

Proof. In this case, $P(C', C)$ and $P(C, C'')$ are upper triangular matrices. Hence E is decomposable by Corollary 3.3 in [3].

Q. E. D.

COROLLARY 2.6. Assume that one of following conditions

(1) $a_1 > a_2 > a_3$ and $c'_1 > c'_2 > c'_3$.

(2) $\{b_1, b_2, b_3\}$ and $\{c_1, c_2, c_3\}$ are the same order.

holds. Then E is decomposable.

LEMMA 2.7. Assume that there exist h such that

(1) $b_h < b_i$ for every $i \neq h$.

(2) $c_h < c_i$ for every $i \neq h$.

(3) $c_h \neq c'_h$.

Then P in (m, P) does not exist.

Proof. Since b_h is the smallest number, we have

$$P(C', C)_{hj} \begin{cases} \neq 0 & \text{for } j = h, \\ = 0 & \text{for } j \neq h. \end{cases}$$

Furthermore, since $P(C, C'')$ is an upper triangular matrix, we have

$$P(C', C'')_{hh} = (P(C', C)P(C, C''))_{hh} \neq 0.$$

Hence $\varphi_0(\xi(C')_h) \geq \varphi_0(\xi(C'')_h)$. This is a contradiction since c_h is the smallest number and $c_h \neq c'_h$

Q. E. D.

COROLLARY 2.8. Assume that there exist h such that

(1) $b_h > b_i$ for every $i \neq h$.

(2) $c_h > c_i$ for every $i \neq h$.

(3) $c_h \neq c'_h$.

Then P in (m, P) does not exist.

LEMMA 2.9. Assume that there exist h such that

(1) $b_h < b_i$ for every $i < h$.

(2) $c_h < c'_h$.

Then P in (m, P) dose not exist.

Proof. By the condition (1) we have

$$P(C', C)_{hj} \begin{cases} \neq 0 & \text{for } j = h. \\ = 0 & \text{for } j < h. \end{cases}$$

Since $P(C, C'')$ is an upper triangular matrix we have $P(C', C'')_{hh} \neq 0$. This is a contradiction to (2).

Q. E. D.

COROLLARY 2.10. Assume that there exist h such that

(1) $b_h > b_i$ for every $i > h$.

(2) $c'_h > c_h$.

Then P in (m, P) does not exist.

COROLLARY 2.11. Assume that $c_1 < c'_1$ or $c_3 > c'_3$. Then P in (m, P) does not exist.

Proof. We apply Lemma 2.9 or Corollary 2.10 for the case $h = 1$ or $h = 3$.

Q. E. D.

Now we can caluculate m in (m, P) for an indecomposable T-equivariant vector bundle of rank three on $\mathbf{P}^2$. Furtheremore we caluculate P in (m, P) and consider the indecomposable case then we have :

THEOREM 2.12. Let $E = E(m, P)$ be an indecomposable T-equivariant vector bundle of rank three on $\mathbf{P}^2$. Then (m, P) is T-equivalent to one of the following table.

<table>
<tr><th>(a_1, a_2, a_3)</th><th>(b_1, b_2, b_3)</th><th>(c_1, c_2, c_3)</th><th>(c'_1, c'_2, c'_3)</th><th>$P(C', C)$</th><th>$P(C, C'')$</th></tr>
<tr><td rowspan="7">(l, m, s)</td><td rowspan="5">(s, m, l)</td><td>(m, l, s)</td><td>(s, m, l)</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td>(l, s, m)</td><td>(s, m, l)</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td rowspan="3">(l, m, s)</td><td>(s, m, l)</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 1 & b & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & -b & -c \\ 0 & 1 & -ac \\ 0 & 0 & 1 \end{pmatrix}$
$c=(1-ab)^{-1}$</td></tr>
<tr><td>(s, l, m)</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td>(m, s, l)</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td>(s, l, m)</td><td>(l, s, m)</td><td>(s, m, l)</td><td>$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
<tr><td>(m, s, l)</td><td>(m, l, s)</td><td>(s, m, l)</td><td>$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$</td><td>$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$</td></tr>
</table>

Here s (m, l resp.) means the smallest (middle, largest resp.) integer.

REFERENCES

[1] T. Kaneyama : On equivariant vector bundles on an almost homogeneous variety. Nagoya Math. J. 57 (1975), 65-86.

[2] T. Kaneyama : Torus-equivariant vector bundles on projective spaces, to appear.

[3] T. Oda : Torus embeddings and applications. Tata Institute of Fundamental Research 57.

[4] T. Oda : Convex bodies and algebraic geometry (in Japanese). Kinokuniya Syoten.

[5] T. Oda and K. Miyake : Almost homogeneous algebraic variety under torus action. Manifolds Tokyo 1973, 373-381.