# The Diagonalization of Derivations

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## §1. Introduction

Let $(A, M)$ be a complete local ring containing a field of characteristic zero and let $K$ be a coefficient field of $A$. Let $\delta$ be a $K$-derivation of $A$ with $\delta M \subset M$. Then $\delta$ induces a $K$-linear map $\overline{\delta} : M/M^2 \to M/M^2$. $\delta$ is said to be diagonalizable if there exist a minimal basis $\{x_1, \cdots, x_n\}$ of $M$ and a subset $\{\alpha_1, \cdots, \alpha_n\}$ of $K$ such that $\delta x_i = \alpha_i x_i$ for each $i$. If $\delta$ is diagonalizable, $\overline{\delta}$ is of course diagonalizable as a $K$-linear map. In this note we want to consider the converse. Our main result is THEOREM 4 which gives a sufficient condition of diagonalizability of $\delta$ in the case $K = \mathbf{C}$.

## §2. Reiffen's Theorem

Let $\alpha_1, \cdots, \alpha_n$ be positive real numbers and let $H$ be the semigroup they generate. There is a bijection $\varphi : \mathbf{N} \to H$ such that $\varphi(0) = 0$ and $\varphi(i) < \varphi(i+1)$ for each $i$. Let $A = \mathbf{C}[\![X_1, \cdots, X_n]\!]$ be a formal power series ring over $\mathbf{C}$. For a monomial $M = X_1^{e_1} \cdots X_n^{e_n}$ of $A$, we put $\deg(M) = \sum_{i=1}^{n} \alpha_i e_i$.

Let $I_k$ be the ideal of $A$ generated by all monomials of degree $\geq \varphi(k)$. Then we have a filtration $\mathbf{F} : I_1 \supset I_2 \supset \cdots$. Note that the topology defined by $\mathbf{F}$ coincides with the $m_A$-adic topology.

The following lemma was proved by Reiffen when all $\alpha_i$ are positive integers ([1] Satz 5). Actually this is valid with $\alpha_i$ positive real numbers instead of positive integers. For completeness we restate it with a simple proof.

LEMMA 1. *With the same notation and assumption as above, let $\delta$ be a $\mathbf{C}$-derivation of $A$. Assume that there exist $k_1, \cdots, k_n \in \mathbf{N}$ such that $\delta X_i \equiv \alpha_i X_i \bmod I_{k_i}$ and $\varphi(k_i) > \alpha_i$ for $i = 1, \cdots, n$. Then there exists a minimal basis $\{Z_1, \cdots, Z_n\}$ of $m_A$ such that $\delta Z_i = \alpha_i Z_i$ $(1 \leq i \leq n)$.*

*Proof.* We may assume $\alpha_1 \leq \cdots \leq \alpha_n$. If $k_i > k_{i+1}$ we have $\delta X_i \equiv \alpha_i X_i \bmod I_{k_{i+1}}$ and $\varphi(k_{i+1}) >$

$\alpha_{i+1} \geq \alpha_i$. Thus replacing $k_i$ by $k_{i+1}$, if necessary, we may also assume $k_1 \leq \cdots \leq k_n$, hence $I_{k_1} \supset \cdots \supset I_{k_n}$. For simplicity we fix one $i$ $(1 \leq i \leq n)$ and put $\alpha = \alpha_i$, $X = X_i$, $k = k_i$ and $r = \varphi(k_i)$. By the assumption we have $\delta X \equiv \alpha X + Y \mod I_{k+1}$ where $Y$ is a homogeneous polynomial of degree $r$. Put $X^{(1)} = X + (\alpha - r)^{-1} Y$. Then it is easy to see that $\delta X^{(1)} = \alpha X^{(1)} \mod I_{k+1}$. Hence we obtain inductively a sequence $\{X^{(j)}\}_{j \geq 1}$ such that $\delta X^{(j)} \equiv \alpha X^{(j)} \mod I_{k+j}$ and $X^{(j)} \equiv X^{(j+1)} \mod I_{k+j}$. If we put $Z = \varinjlim_j X^{(j)}$ then it is clear that $\delta Z = \alpha Z$. It remains to show that $\{Z_1, \cdots, Z_n\}$ is a minimal basis of $m_{\mathrm{A}}$. Note that $X_i \notin I_j$ for $j > k_i$ by the assumption $\alpha_i < \varphi(k_i)$. Thus, by the construction of $Z_i$, we have

$$Z_i \equiv X_i + C_{i,i+1} X_{i+1} + \cdots + C_{i,n} X_n \mod m_{\mathrm{A}}^2 \quad (1 \leq i \leq n)$$

where $C_{i,j} \in \mathbf{C}$. This shows that $\{Z_1, \cdots, Z_n\}$ is a minimal basis of $m_{\mathrm{A}}$.

COROLLARY 2. *Let $\nu$ be a positive integer. Assume that*

(1) $\delta X_i \equiv \alpha_i X_i \mod m_{\mathrm{A}}^{\nu}$ *and*

(2) $\nu > \max\{\alpha_i / \alpha_j \mid 1 \leq i, j \leq n\}$.

*Then $\delta$ is diagonalizable.*

*Proof.* Immediate from LEMMA 1.

## §3. Main Result

Let $K$ be a field and let $S = \{\alpha_1, \cdots, \alpha_n\}$ be a finite subset of $K$. We consider the following condition on $S$.

(*) For each $i$, if $\sum_{j=1}^{n} m_j \alpha_j = \alpha_i$ where $m_1, \cdots, m_n$ are non-negative integers, then $m_i = 1$ and $m_j = 0$ for $j \neq i$.

*Remarks.* (1) If $S$ is a linearly independent set over $Z$ then $S$ obviously satisfies the condition (*).

(2) Let $H$ be the semigroup generated by $S$. If $S$ satisfies the condition (*) then $S$ is a set of minimal generators of $H$. The converse is also true if all $\alpha_i$ are positive real numbers.

LEMMA 3. *Let $K$ be a field of characteristic zero and $S = \{\alpha_1, \cdots, \alpha_n\}$ a finite subset of $K$. Let $A = K[\![X_1, \cdots, X_n]\!]$ be a formal power series ring over $K$ and $\delta$ a $K$-derivation of $A$. Assume that*

(1) $\delta X_i \equiv \alpha_i X_i \bmod m_A^\nu$ *($\nu \geq 2$, $1 \leq i \leq n$) and*

(2) *$S$ satisfies the condition (*).*

*Then, for any sequence $\beta_1, \cdots, \beta_n$ of elements of $K$, there exists a $K$-derivation $\partial$ of $A$ such that $\partial\delta = \delta\partial$ and $\partial X_i \equiv \beta_i X_i$ mod $m_A^\nu$ ($1 \leq i \leq n$).*

*Proof.* It is enough to show that there is a sequence $\{\partial_j\}_{j\geq 1}$ satisfying the following :

(a) $\partial_j \in m_A^j \operatorname{Der}_k(A)$ and

(b) if we put $\partial^{(j)} = \partial_1 + \cdots\cdots + \partial_j$, then for every $i$, $\partial^{(j)} X_i \equiv \beta_i X_i \bmod m_A^\nu$ and $(\partial^{(j)}\delta - \delta\partial^{(j)})(X_i) \equiv 0 \bmod m_A^{j+1}$.

We choose $\partial_j$ by induction. Put $\partial_1 = \sum_{i=1}^{n} \beta_i X_i \frac{\partial}{\partial X_i}$ and $\partial_2 = \cdots = \partial_{\nu-1} = 0$. Let $j \geq \nu$ and assume

$$(\partial^{(j)}\delta - \delta\partial^{(j)})\,(X_i) \equiv \sum_{e_1+\cdots\cdots+e_n = j+1} c_{i,e_1,\ldots,e_n} X_1^{e_1} \cdots X_n^{e_n} \bmod m_A^{j+2} \quad (c_{i,e_1,\ldots,e_n} \in K).$$

Since $\alpha_1 e_1 + \cdots + \alpha_n e_n - \alpha_i \neq 0$ by the condition (*), we can write

$$\partial_{j+1} = \sum_{i=1}^{n} \Big( \sum_{e_1,\cdots,e_n} (\alpha_1 e_1 + \cdots + \alpha_n e_n - \alpha_i)^{-1} c_{i,e_1,\ldots,e_n} X_1^{e_1} \cdots X_n^{e_n} \Big) \frac{\partial}{\partial X_i}$$

It can be seen by easy computation that $\{\partial_j\}_{j\geq 1}$ satisfies the requirement.

Combining COROLLARY 2 and LEMMA 3, we can prove the following, which is our main result.

THEOREM 4. *Let $(A, M)$ be a complete local ring containing* $\mathbf{C}$ as a coefficient field. Let $\delta$ be a $\mathbf{C}$-*derivation of $A$. Assume that there exist real numbers $\alpha_1, \cdots, \alpha_n$ and a minimal basis $\{x_1, \cdots, x_n\}$ of $M$ such that*

(1) $\delta x_i \equiv \alpha_i x_i \bmod M^2$ *and*

(2) *the set $\{\alpha_1, \cdots, \alpha_n\}$ satisfies the condition (*).*

*Then there exists a minimal basis $\{y_1, \cdots, y_n\}$ of $M$ such that $\delta y_i = \delta_i y_i$ for $i = 1, \cdots, n$.*

*Proof.* Let $S = \mathbf{C}[\![X_1, \cdots, X_n]\!]$ be a formal power series ring and let $f : S \to A$ be a surjective $\mathbf{C}$-homomorphism with $f(X_i) = x_i$. Then there is a $\mathbf{C}$-derivation $D$ of $S$ suchthat $\delta \circ f = f \circ D$. If $D$ is diagonalizable so is $\delta$. Since $DX_i \equiv \alpha_i X_i \bmod m_S^2$, we may assume $A = S$. By the condition (*), we may assume $\alpha_1 < \cdots < \alpha_n$. We take a real number $\beta$ such that $\beta > \alpha_n - \alpha_1$. Put $\gamma = \alpha_n + \beta$ and $\beta_i = \gamma - \alpha_i (1 \leq i \leq n)$. Then we have $0 < \beta_n < \cdots < \beta_1$ and $\beta_1/\beta_n < 2$. By LEMMA 3, we have a $\mathbf{C}$-derivation $\partial$ of $A$ such that $\partial x_i \equiv \beta_i x_i \bmod M^2$ and $\partial\delta = \delta\partial$. We put $\Delta = \delta + \partial$. Since $\partial x_i \equiv \beta_i x_i$ mod $M^2$ and $\Delta x_i \equiv \gamma x_i \bmod M^2$, $\partial$ and $\Delta$ are both diagonalizable by COROLLARY 2. Moreover, since

$ə\Delta=\Delta ə$, $ə$ and $\Delta$ are simultaneously diagonalizable, that is, there exists a minimal basis $\{y_1, \cdots, y_n\}$ of $M$ such that $ə y_i=\beta_i y_i$ and $\Delta y_i=\gamma y_i (1\leq i\leq n)$. Hence we have $\delta y_i=\alpha_i y_i$.

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