Line Bundles on a Hyperelliptic Curve

by

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§1 Introduction

Let X be a hyperelliptic curve of genus $g \geqq 2$ with a double covering $f: X \to P$ where P means the one-dimensional projective space P_k^1 over the algebraic closed field k. This double covering f is defined by the linear system of dimension one and degree two on X. Let L be a line bundle on X. Since f is a finite morphism of degree two, the direct image f_*L of L is a vector bundle of rank two on P. Furthermore since every vector bundle on P is direct sum of line bundles, we can write $f_*L = \mathcal{O}_P(a) \oplus \mathcal{O}_P(b)$ $(a \geqq b)$. In this paper we study the decomposition of f_*L.

§2 Notations and some preliminary results

DEFINITION 2.1 A curve X is called hyperelliptic if the genus g of X is $g \geqq 2$ and if there exists a finite morphism $f: X \to P$ of degree two where P is the one-dimensional projective space P_k^1.

In this paper we denote the one dimensional projective space by P. Since we use properties of line bundles on P and their cohomological groups, we state some results on a projective space P. This is a part of Serre's theorem that is stated on general projective space.

PROPOSITION 2.2 *Let P be a one-dimensional projective space. Then*

(1) $H^0(P, \mathcal{O}_P(r)) = 0$ *if* $r \leqq -1$

(2) $H^1(P, \mathcal{O}_P(r)) = 0$ *if* $r \geqq -1$

(3) $\dim H^0(P, \mathcal{O}_P(r)) = r+1$ *if* $r \geqq 0$

(4) $\dim H^1(P, \mathcal{O}_P(r)) = -r-1$ *if* $r \leqq -2$

§3 Direct image of line bundles

We state at first the dimect image f_*L in case of $L=O_X$ and $L=K_X$. This case is completely described. The results are the following proposition.

PROPOSITION 3.1 *Let X be a hyperelliptic curve of genus g $(g\geqq 2)$ and let $f: X\to P$ be a double corvering then*

(1) $f_*O_X=O_P\oplus O_P(-g-1)$

(2) $f_*K_X=O_P(g-1)\oplus O_P(-2)$

where O_X is the structure sheaf of X and K_X is the canonical sheaf of X.

proof Since $f: X\to P$ is a finite morphism of degree two, the direct image f_*L of a line bundle L on a curve X is of the form

$$f_*L=O_P(a)\oplus O_P(b) \qquad (a\geqq b)$$

where a, b are integers. Since f is a finite morphism, we see by using spectral sequence that

$$\dim H^i(X, L)=\dim H^i(P, f_*L) \qquad \text{for } i=0, 1.$$

By using Riemann-Roch theorem to the line bundle L, we have

$$\dim H^0(X, L)-\dim H^1(X, L)=\deg L+1-g.$$

This is a special case of Riemann-Roch. Therefore we have

$$\dim H^0(P, f_*L)-\dim H^1(P, f_*L)=\deg L+1-g.$$

On the other hand

$$\begin{aligned}&\dim H^0(P, f_*L)-\dim H^1(P, f_*L)\\ &\quad=\dim H^0(P, O_P(a))-\dim H^1(P, O_P(a))\\ &\qquad\quad+\dim H^0(P, O_P(b))-\dim H^1(P, O_P(b))\\ &\quad=(a+1)+(b+1)\end{aligned}$$

$$=a+b+2\,.$$

Then we have

$$a+b=\deg L-g-1\,.$$

(1) In case of $L=O_X$.

We know the dimension of cohomological groups of X. Therefore

$$\begin{aligned}\dim H^0(X, O_X) &= \dim H^0(P, f_*O_X)\\ &= \dim H^0(P, O_P(a))+\dim H^0(P, O_P(b))\end{aligned}$$

is equal to one. On the other hand we also know the dimension of cohomological groups on P by Proposition 2.2. So the integers a and b satisfy that $a=0$ and $b\leqq -1$. By using the equality $a+b=g-1$ that is stated above, we have $b=-g-1$. Therefore

$$f_*O_X=O_P\oplus O_P(-g-1).$$

(2) In case of $L=K_X$.

We do the similar way. Since

$$\begin{aligned}\dim H^1(X, K_X) &= \dim H^1(P, f_*K_X)\\ &= \dim H^1(P, O_P(a))+\dim H^1(P, O_P(b))\\ &= \dim H^0(P, O_P(-a-2))+\dim H^0(P, O_P(-b-2))\end{aligned}$$

is equal to one. Again by uning the dimension of cohomological groups on P, we have

$$-b-2=0\,,\ -a-2\leqq -1$$

since $a\geqq b$. In this case, the equality which is stated at first is $a+b=g-3$. This implies that

$$a=g-1\,,\ b=-1\,.$$

Therefore

$$f_*K_X=O_P(g-1)\oplus O_P(-2).$$

The proof has done.

CORORALLY 3.2 $K_X \approx f^* O_P(g-1)$

proof Put $L = f^* O_P(g-1)$. By the projection formula

$$
\begin{aligned}
f_* L &= f_*(O_X \otimes L) \\
&= f_* O_X \otimes O_P(g-1) \\
&= \{O_P \oplus O_P(-g-1)\} \otimes O_P(g-1) \\
&= O_P(g-1) \oplus O_P(-2).
\end{aligned}
$$

If L is not isomorphic to the canonical line bundle K_X, the dimension of a cohomological group $H^0(X, L)$ does not equal to g. On the other hand since $g \geqq 2$ we know that

$$
\begin{aligned}
\dim H^0(X, L) &= \dim H^0(P, f_* L) \\
&= \dim H^0(P, O_P(g-1)) + \dim H^0(P, O_P(-2)) \\
&= g.
\end{aligned}
$$

This is a contradiction. So we have $K_X \approx f^* O_P(g-1)$.

Q.E.D.

Put $M = f^* O_P(1)$. Since M is of degree two and the dimension of $H^0(X, M)$ is two, M defines the double covering $f: X \to P$.

The canonical line bundle K_X determines the morphism h from X to the $(g-1)$-dimensional projective space P^{g-1}. This morphism is called a canonical morphism. This canonical morphism is obtained by f and $(g-1)$*-ple embedding of* P^1 into P^{g-1} so that the diagram

$$
\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "f"'] & P^{g-1} \\
P^1 \arrow[ur] &
\end{tikzcd}
$$

is commutative. This result is clear by using the fact which is stated above. [See 2]

DEFINITION 3.3 Let X be a hyperelliptic curve and let L be a line bundle on X. We define the integer $m(L)$ by

$$m(L)=min\ \{m \mid H^0(X,\ L\otimes M^m)\neq 0\}$$

where $M=f^*O_P(1)$ and $f: X\to P$ is a double covering.

PROPOSITION 3.4 *Let L and N be line bundles on a hyperelliptic curve X of genus g $(g\geqq 2)$. Then*

(1) $-\frac{1}{2}\deg L\leqq m(L)\leqq\frac{1}{2}(g+1-\deg L)$

(2) $m(L\otimes N)\leqq m(L)+m(N)$

proof Before prove the fact (1), we state general results. Let L be a line bundle on X. Riemann-Roch theorem for L

$$\dim H^0(X,\ L)-\dim H^1(X,\ L)=\deg L+1-g$$

implies

$$\dim H^0(X,\ L)\geqq\deg L+1-g.$$

Therefore if $\deg L>g-1$ then $\dim H^0(X,\ L)\geqq 1$. In other words, $H^0(X,\ L)=0$ implies $\deg L\leqq g-1$.

Now we want to prove the fact (1). Since $H^0(X,\ L\otimes M^{m(L)})\neq 0$, the degree of $L\otimes M^{m(L)}$ must be non-negative, *i.e.*

$$\begin{aligned}\deg L\otimes M^{m(L)}&=\deg L+m(L)\deg M\\ &=\deg L+2m(L)\geqq 0.\end{aligned}$$

From the minimality of $m(L)$ such that $H^0(X,\ L\otimes M^{m(L)})\neq 0$, we have

$$H^0(X,\ L\otimes M^{m(L)-1})=0.$$

The fact that is stated above induces the inequality

$$\begin{aligned}\deg L\otimes M^{m(L)-1}&=\deg L+\{m(L)-1\}\deg M\\ &=\deg L+2m(L)-2\leqq g-1.\end{aligned}$$

Combining these two inequalities, we have

$$-\frac{1}{2}\deg L\leqq m(L)\leqq\frac{1}{2}(g+1-\deg L).$$

Now (1) is proved. Next we prove the fact (2). Since $H^0(X,\ L\otimes M^{m(L)})\neq 0$ we have a sequence

$$0 \to O_X \to L \otimes M^{m(L)}.$$

Tensoring $N \otimes M^{m(N)}$ to this sequence, we have

$$0 \to N \otimes M^{m(N)} \to L \otimes N \otimes M^{m(L)+m(N)}$$

On the other hand, since $H^0(X, N \otimes M^{m(N)}) \neq 0$ we have a sequence

$$0 \to O_X \to N \otimes M^{m(N)}$$

From these two exact sequences, we have a diagram

$$\begin{tikzcd} & & 0 \arrow[d] \\ 0 \arrow[r] & O_X \arrow[r] \arrow[dr, dotted, "s"] & N \otimes M^{m(N)} \arrow[d] \\ & & L \otimes N \otimes M^{m(L)+m(N)} \end{tikzcd}$$

In the diagram, the dotted arrow is composed map s. This map s defines the non-zero section of $L \otimes N \otimes M^{m(L)+m(N)}$ Then by the definition of $m(L \otimes N)$, we have

$$m(L \otimes N) \leqq m(L) + m(N).$$

Q.E.D.

PROPOSITION 3.5 *Let X be a hyperelliptic curve of genus g with a double covering $f: X \to P$. Let L be a line bundle on X of degree d. Then*

$$f_* L = O_P(-m(L)) \otimes O_P(d + m(L) - g - 1)$$

where $m(L)$ is the integer defined above.

proof Put

$$f_* L = O_P(a) \oplus O_P(b) \qquad (a \geqq b).$$

Then we know by Riemann-Roch theorem that $a + b = d - g - 1$. Since $H^0(X, L \otimes M^{m(L)}) \neq 0$ and

$$f_*(L\otimes M^{m(L)})=f_*L\otimes O_P(m(L))$$
$$=O_P(a+m(L))\oplus O_P(b+m(L)),$$

we have

$$a+m(L)\geqq 0\,.$$

On the other hand, the same argument is used. Since $H^0(X,\ L\otimes M^{m(L)-1})=0$ by the definition of $m(L)$, we have

$$a+m(L)-1\leqq -1\,.$$

These two inequalities means

$$a+m(L)=0\,.$$

Therefore

$$b=d+m(L)-g-1.$$

Then we have

$$f_*L=O_P(-m(L))\oplus O_P(d+m(L)-g-1).$$

Q.E.D.

PROPOSITION 3.6 *Let L be a line bundle of degree d on the hyperelliptic curve X of genus g. If* deg $L\leqq g-1$ *then*

$$\dim\ H^0(X,\ L)\leqq\frac{1}{2}d+1\,.$$

proof Using the inequality of Proposition 3.4, we have

$$d+m(L)-g-1\leqq\frac{1}{2}(d-g-1)\leqq -1\,.$$

Since

$$\dim\ H^0(X,\ L)=\dim\ H^0(P,\ f_*L)$$

and

$$f_*L=O_P(-m(L))\oplus O_P(d+m(L)-g-1)$$

We have

$$\dim H^0(X, L) = -m(L) + 1 .$$

Again using the inequality of Proposition 3.4, we have

$$\dim H^0(X, L) \leqq \frac{1}{2}d + 1 .$$

Q.E.D.

REMARK 3.7 There is Clifford's theorem that is a bound on dimension of a complete linear system of some divisor on a curve. [See 2] We state precisely. Let D be an effective special divisor on the curve X. Then $\dim |D| \leqq \frac{1}{2}\deg D$. Above proposition is a case of hyperelliptic curve. But Clifford's theorem does not implies this proposition.

§4 Line bundles on a hyperelliptic curve of genus two

In this section we consider the case that X is a hyperelliptic curve of genus two. We embed X into its Jacobian variety J. Let x be a point of X. Then the point x is considered as the point of J. Then we write a corresponding line bundle on X by $O_X(x)$.

PROPOSITION 4.1 *Let X be a hyperelliptic curve of genus two and let $f : X \to P$ be a double covering. Let L be a line bundle on X. Then*

(1) $\deg L = 0$

(a) *if* $L \approx O_X$ *then* $f_*L = O_P \oplus O_P(-3)$,

(b) *if* $L \not\approx O_X$ *then* $f_*L = O_P(-1) \oplus O_P(-2)$,

(2) $\deg L = 1$

(c) *if* $L \approx O_X(x)$ *for some* x *in* X *then* $f_*L = O_P \oplus O_P(-2)$,

(d) *if* $L \not\approx O_X(x)$ *for any* x *in* X *then* $f_*L = O_P(-1) \oplus O_P(-1)$,

(3) $\deg L = 2$

(e) *if* $L \approx K_X$ *then* $f_*L = O_P(1) \oplus O_P(-2)$,

(f) *if* $L \not\approx K_X$ *then* $f_*L = O_P \oplus O_P(-1)$.

proof (a) and (b) are already proved in Proposition 3.1. The idea of proof is similar to that of Proposition 3.1. We know the dimension of the cohomological group $H^0(X, L)$ for every L.

Therefore the results are easily obtained.

Q.E.D.

REMARK 4.2 In the case of a hyperelliptic curve of genus two, we know the value $m(L)$ by tonsoring M^n to L for suitable n.

COROLLALLY 4.3 $K \approx f^* O_P(1)$

REMARK 4.4 In the case of a hyperelliptic curve of genus two, the canonical line bundle K and M are isomorphic. Therefore the canonical line bundle K defines the double covering $f: X \to P$.

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