TORUS ACTIONS ON $GL_r(k[x_1, x_2, \cdots x_n])$

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§ 1 Introduction and notations

Let k be an algebraically closed field of arbitrary characteristic. Let T be an n-dimensional algebraic torus, *i.e.* $T = G_m \times \cdots \times G_m$ (n-times), where $G_m = \mathrm{Spec}\ (k\ [t, t^{-1}])$ is the multiplicative group. Let $A = k\,[x_1, \cdots, x_n]$ a polynomial ring. T acts eeffectively and semi-linearly on A. Let (E, φ_t) be a T-linearized equivariant vector bundle on $X = \mathrm{Spec}\ A$. (See [2]). Let $E = \tilde{M}$ where M is an A-module. Then M is a free module by [2] . By the definition of T-linearized equivariant vector bundle (E, φ_t), $\{\varphi_t\}$ satisfy the condition $\varphi_{tt'} = \varphi_{t'} \cdot t'^{\,*}\varphi_t$. These correspond $\{\rho_t\}$ in $GL_r(A)$ such that

$$\rho_{tt'}(x_1, x_2, \cdots, x_n) = \rho_{t'}(x_1, x_2, \cdots, x_n)\ \rho_t(t'x_1, t'x_2, \cdots, t'x_n)$$

for every t, t' in T where $r = \mathrm{rank}\ (E)$. We consider $\{\rho_t\}$ in the following section.

§ 2 Group cohomology

Let $\Xi = \Xi(T) = Hom_{k\text{-}gr}\,(T, G_m)$ be the additive group of characters of T. Ξ is a free $\boldsymbol{Z}$-module of rank $n = \dim T$.

THEOREM 2.1 *Let T be an n-dimensional torus. T acts effectively and semi-linearly on $A = k\ [x_1, x_2, \cdots, x_n]$. Then $H^1_{gr}\,(T, GL_r\,(A)) = \Xi^{+r}$.*

To prove this theorem, we interpret the theorem down to earth, that is to say we may prove the following theorem 2.2.

THEOREM 2.2 *Let $\rho(t_1, t_2, \cdots, t_n\ ;\ x_1, x_2, \cdots, x_n)$ be a matrix in $GL_r\,(k\,[t_1, t_1^{-1}, t_2, t_2^{-1}, \cdots, t_n, t_n^{-1}, x_1, x_2, \cdots, x_n])$ that satisfies the condition*

$$\begin{aligned}&\rho\,(t_1t_1', t_2t_2', \cdots, t_nt_n'\ ;\ x_1, x_2, \cdots, x_n)\\ &= \rho\,(t_1, t_2, \cdots, t_n\ ;\ x_1, x_2, \cdots, x_n)\ \rho\,(t'_1, t'_2, \cdots, t_n'\ ;\ t_1x_1, t_2x_2, \cdots, t_nx_n).\end{aligned}$$

Then there exists a matrix $\sigma(x_1, x_2, \cdots, x_n)$ in the $GL_r\,(k[x_1, x_2, \cdots, x_n])$ such that

$$\rho\,(t_1, t_2, \cdots, t_n\ ;\ x_1, x_2, \cdots, x_n) = \sigma(x_1, x_2, \cdots, x_n)\ E\sigma(t_1x_1, t_2x_2, \cdots, t_nx_n)^{-1}.$$

where E is a diagonal matrix whose diagonal elements are of the form $t_1^{p_1}t_2^{p_2}\cdots t_n^{p_n}$ (p_i : integers).

To prove this theorem, we prepare some results. At first we prove the following proposition which is a special case of theorem 2.2.

PROPOSITION 2.3 *Let* $\rho_1(t\ ;\ x)$ *be a matrix in* $GL_r\ (k\ [t,\ t^{-1},\ x])$ *such that*

$$\rho_1(tt'\ ;\ x) = \rho_1(t\ ;\ x)\ \rho_1(t'\ ;\ tx).$$

Then there exists a matrix $\sigma_1(x)$ *in* $GL_r\ (k\ [x])$ *such that*

$$\rho_1(t\ ;\ x) = \sigma_1(x)\ E_1\sigma_1(tx)^{-1}$$

where E_1 *is a diagonal matrix whose diagonal elements are of the form* t^{p_i} $(p_i\ ;\ integers)$.

Proof. In the relation of ρ_1 we put $x=1$ then we have

$$\rho_1(tt'\ ;\ 1) = \rho_1(t\ ;\ 1)\ \rho_1(t'\ ;\ t).$$

we write t, x in place of t', t respectively then we have

$$\rho_1(t\ ;\ x) = \rho_1(x\ ;\ 1)^{-1}\rho_1(tx\ ;\ 1).$$

But this is not defined at $x=0$. For every S in $GL_r\ (k)$, we can write

$$\rho_1(x\ ;\ 1)^{-1}\ S = \sigma_1(x)\ E_1(x)^{-1}$$

where $\sigma_1\ (x)$ in $M_r\ (k\ [x])$, det $\sigma_1\ (x) = x^d\ (d \geqq 0)$ and $E_1(x)$ is a diagonal matrix whose diagonal elements are of the from x^{α_i} (α_i : integers). We take S in $GL_r\ (k)$ such that d is minimal. Then

$$\rho_1(t\ ;\ x) = \sigma_1(x)\ E_1(t)\ \sigma_1(tx)^{-1}.$$

If $d=0$ then the proposition is proved. So we may assume $d \geqq 1$. Furthermore we may assume that

$$E_1(t) = diag\ (\underbrace{t^{\alpha_1},\ \cdots,\ t^{\alpha_1}}_{p_1},\ \underbrace{t^{\alpha_2},\ \cdots,\ t^{\alpha_2}}_{p_2},\ t^{\alpha_3},\ \cdots)$$

where $\alpha_i \neq \alpha_j$ if $i \neq j$. $diag\ (a_1,\ a_2,\ \cdots,\ a_n)$ means a diagonal matrix whose diagonal elements are $a_1,\ a_2,\ \cdots,\ a_n$.

Since $d \geqq 1$, there is a component of $\sigma_1\ (tx)^{-1}$ which has a denominator, say, it is in the m-th column. Then $\sigma_1\ (tx)^{-1}$ is of the form

$$\sigma_1\ (tx)^{-1} = \begin{array}{c} \begin{array}{ccc} \qquad & \overbrace{\qquad}^{m\text{-th}} & \qquad \end{array} \\ \begin{pmatrix} \cdots\cdots & \dfrac{a_1}{(tx)^q}+\cdots & \cdots\cdots \\ & \vdots & \\ & \vdots & \\ \cdots\cdots & \dfrac{a_r}{(tx)^q}+\cdots & \cdots\cdots \end{pmatrix} \end{array} \qquad (a_i\ \epsilon k)$$

where at least one of a_i is not zero. There is a scalar matrix H such that

$$H\ \sigma_1\ (tx)^{-1}=\begin{pmatrix} \cdots\cdots & \dfrac{a'_1}{(tx)^q}+\cdots & \cdots\cdots \\ \cdots\cdots & 0\ \ +\cdots & \cdots\cdots \\ & \vdots & \\ \cdots\cdots & 0\ \ +\cdots & \cdots\cdots \\ \cdots\cdots & \dfrac{a'_{p1+1}}{(tx)^q}+\cdots & \cdots\cdots \\ \cdots\cdots & 0\ \ +\cdots & \cdots\cdots \\ & \vdots & \end{pmatrix}$$

where at least one of $a'_1,\ a'_{p1+1},\ \cdots$ is not zero. Since H is of the form

$$\begin{pmatrix} \boxed{*} & & & \\ & \boxed{*} & & 0 \\ & & \ddots & \\ 0 & & & \boxed{*} \end{pmatrix}\begin{matrix} \}\ p_1 \\ \}\ p_2 \\ \\ \\ \end{matrix}$$

we have $HEH^{-1}=E$. Therefore we may replace $\sigma_1\ (tx)^{-1}$ by $H\ \sigma_1\ (tx)^{-1}$ for

$$\begin{aligned} \sigma_1\ (x)\ E_1\ (x)\ \sigma_1\ (tx)^{-1} &= (\sigma_1\ (x)\ H^{-1})\ HEH^{-1}\ (H\sigma_1\ (tx)^{-1}) \\ &= (\sigma_1\ (x)\ H^{-1})\ E\ (H\ \sigma_1\ (tx)^{-1}). \end{aligned}$$

Let $(i,\ j)$-component of $\sigma_1\ (x)$ be $b_{ij}^{(0)}+b_{ij}^{(1)}x+b_{ij}^{(2)}x^2+\cdots$. Since $\rho_1\ (t\ ;\ x)$ is in $GL_r(k[t,\ t^{-1},\ x])$, by expanding

$$\rho_1(t\ ;\ x)=\rho_1(x\ ;\ 1)^{-1}\rho_1(tx\ ;\ 1)=\sigma_1(x)E_1(t)\sigma_1(tx)^{-1},$$

we know that $(tx)^{-q}\ (b_{i1}^{(0)}\ a'_1t^{\alpha 1}+b_{ip1+1}^{(0)}\ a'_{p1+1}t^{\alpha 2}+\cdots)$ must be in $x^{-(q-1)}k\ [t,\ t^{-1},\ x]$ for every i.

Since there is a non-zero a'_l and since α_k are mutually different, $b_{il}^{(0)}$ must be zero for every i. Therefore

$$\sigma_1\ (x)\ =\tilde{\sigma}_1\ (x)\begin{array}{c} \begin{matrix} & & l & & \end{matrix} \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & 0 & \\ & & 1 & & \\ & & & x & \\ & & & & 1 \\ & 0 & & & \ddots \\ & & & & & 1 \end{pmatrix} \end{array}$$

where $\tilde{\sigma}_1\ (x)$ in $M_n\ (k\ [x])$ and det $\tilde{\sigma}_1=$ (det σ_1) x^{-1}. This contradicts the minimalty of d. So d must be zero. Hence this σ_1 satisfies proposition 2.3. Q.E.D.

We next prove the key proposition but elementary.

PROPOSITION 2.4 *Let* $A(x)=(a_{ij}\ (x))$ *be in* $GL_r\ (k[x])$, $E=diag\ (x^{\alpha_1},\ x^{\alpha_2},\ \cdots,\ x^{\alpha_r})$ *and* $\tilde{E}=diag\ (x^{\beta_1},\ x^{\beta_2},\ \cdots,\ x^{\beta_r})$ $(\alpha_i,\ \beta_j$; *integers). If* $E\ A(x)\ \tilde{E}^{-1}=A(1)$ *then there is a scalar matrix* H *such that* $\tilde{E}=HEH^{-1}$.

Furthermore there is a scalar matrix $\tilde{H}$ such that

$$\tilde{H}\,A(x)\,\tilde{H}^{-1}=\begin{pmatrix} * & & & \\ & \boxed{*} & a_{ij}\,(1)\;x^{e_{ij}} & \\ & & \ddots & \\ 0 & & & \boxed{*} \end{pmatrix}$$

where e_{ij} is determined by α_k.

Proof. Since $\det E\ A(x)\ \tilde{E}^{-1}=\det \mathrm{A}(1)$ is in k^*, $\sum_{i=1}^{r}\alpha_i=\sum_{j=1}^{r}\beta_j$. Furthermore since

$$E\ A(x)\ \tilde{E}^{-1}=(a_{ij}\ (x)\ x^{\alpha i-\beta j})=A(1)$$

if $\alpha_i>\beta_j$ then $a_{ij}\ (x)=0$ and if $\alpha_i\leqq\beta_j$ then $a_{ij}\ (x)=a_{ij}\ (1)\ x^{\beta_j-\alpha_i}$.

For any j, if $\beta_j\neq\alpha_i$ for every i, then by above remark, elements of the j-column of $\mathrm{A}(x)$ are of the form $a_{ij}(1)\ x^{\beta_j-\alpha_i}$ or 0. So by expanding $A(x)$ with respect to the j-column $\det A(x)$ is not in k^*. Hence $\beta_j=\alpha_i$ for some $i=i(j)$. If $\alpha_{m1}=\cdots=\alpha_{mp}=\beta_{n1}\cdots=\beta_{mq}$ and $p\neq q$ then by the Laplace expansion of $A(0)$, we have $\det A(x)=\det A(0)=0$. Hence $p=q$. Therefore there is a scalar matrix H such that $\tilde{E}=HEH^{-1}$. The second assertion is clear. Q.E.D.

Proof of THEOREM 2.2. We prove by induction on n. When $n=1$, by PROPOSITION 2.3, the theorem is true. We assume that the theorem is true for $n-1$. Then there is a matrix $\sigma\ (x_1, x_2, \cdots, x_n)$ in $GL_r\ (k(x_n)\ [x_1, x_2, \cdots, x_{n-1}])$ such that

$$\begin{aligned}&\rho\ (t_1, t_2, \cdots, t_{n-1}, 1\ ;\ x_1, x_2, \cdots, x_n)\\ &\quad=\sigma\ (x_1, x_2, \cdots, x_n)\ E\ \sigma\ (t_1x_1, t_2x_2, \cdots, t_{n-1}x_{n-1}, x_n)^{-1}\end{aligned}$$

where E is a diagonal matrix whose diagonal elements are of the form $t_1^{\alpha_{j1}}t_2^{\alpha_{i2}}\cdots t_{n-1}^{\alpha_{in-1}}$ (α_{ij} : integers). From now, we write $(x_1, x_2, \cdots, x_{n-1})$, $(t_1, t_2, \cdots, t_{n-1})$, $(t_1x_1, t_2x_2, \cdots, t_{n-1}x_{n-1})$, $t_1^{\alpha_{i1}}t_2^{\alpha_{i2}}\cdots t_{n-1}^{\alpha_{in-1}}$ by x, t, tx, t^{α_j} for convinient. We can take $\det\sigma\,(x, x_n)=\det\sigma\,(x_1, x_2, \cdots, x_n)$ is in k^*. $\sigma\ (x, x_n)$ must be not defined at finite number of points $f=f_1, f_2, \cdots, f_p$ of $\mathrm{Spec}\,(k\ [x_n])$. So we take this $\sigma\ (x, x_n)$ such that the number of points $f_1, f_2, \cdots, f_p$ at which $\sigma\ (x, x_n)$ is not defined is minimal. We may regard that f_i are irreducible monic polynomials in $k\ [x_n]$. By multiplying some diagonal matrix in $GL_r\ (k\ [x_n])$ from right to $\sigma\,(x, x_n)$ we may assume that $\det\sigma\ (x, x_n)=cf^d$ where c in k^* and $\sigma\ (x, x_n)$ is defined at f and d is minimal positive integer.

Furthermore we can take E of the form

$$diag(\underbrace{t^{\alpha_1}, \cdots, t^{\alpha_1}}_{p_1}, \underbrace{t^{\alpha_2}, \cdots, t^{\alpha_2}}_{p_2}, \cdots)$$

where $\alpha_i\neq\alpha_j$ if $i\neq j$. If $d\geqq 1$ then there is a component of $\sigma\ (tx, x_n)^{-1}$ which has a denominator and is in the m-th column. Let (i, m)-component of $\sigma\ (tx, x_n)^{-1}$ be $\tau_i f^{-b}+\tau_i^{(1)}f^{-b}x_n+\cdots$ where τ_i in $k\ [x, f_2^{-1}, \cdots, f_p^{-1}]$ and $f\nmid\tau_i$. Furthermore we write $\tau_i=a_i^{(0)}+a_i^{(1)}x+\cdots$ where $a_i^{(j)}$ are in $k\ [f_2^{-1}, \cdots, f_p^{-1}]$ and at least one of $a_i^{(0)}$ is not zero because $\det\sigma\ (tx, x_n)^{-1}$ is independent of x.

So there is a matrix H *in* $GL_r(k(f_2, \cdots, f_p^{-1}))$ such that

$$H\sigma(tx, x_n)^{-1} = \begin{pmatrix} \cdots & \bar{a}_1 f^{-b} & + \cdots & \cdots \\ \cdots & 0 & + \cdots & \cdots \\ & \vdots & & \\ \cdots & 0 & + \cdots & \cdots \\ \cdots & \bar{a}_{p_1+1} f^{-b} & + \cdots & \\ \cdots & 0 & + \cdots & \cdots \end{pmatrix}$$

where $\bar{a}_1, \bar{a}_{p_1+1}, \cdots$ are in $k(f_2^{-1}, \cdots, f_p^{-1})$ and at least one of them is not zero. And we way assume that $f \nmid \bar{a}_j$ for every j. Since we can take H of the form

$$\begin{pmatrix} \boxed{*} & & & \\ & \boxed{*} & & 0 \\ & & \ddots & \\ 0 & & & \boxed{*} \end{pmatrix} \begin{matrix} \} \ p_1 \\ \} \ p_2 \\ \\ \\ \end{matrix}$$

$HEH^{-1} = E$. Therefore we may replace $\sigma(tx, x_n)^{-1}$ by $H\ \sigma(tx, x_n)^{-1}$ for

$$\sigma(x, x_n)\ E\ \sigma(tx, x_n)^{-1} = (\sigma(x, x_n)\ H^{-1})\ E\ (H\sigma(tx, x_n)^{-1}).$$

Let $\sigma(x, x_n) = (\sigma_{ij}(x, x_n))$. Since $\rho(x, 1\ ;\ x, x_n)$ is in $GL_r(k[t_1, t_1^{-1}, \cdots, t_{n-1}, t_{n-1}^{-1}, x_1, \cdots, x_n])$, by expanding

$$\rho(t, 1\ ;\ x, x_n) = \sigma(x, x_n)\ E\ \sigma(tx, x_n)^{-1},$$

we know that

$$f^{-b}(\sigma_{ij}(x, x_n)\ \bar{a}_1 t^{\alpha_1} + \sigma_{ip_1+1}\bar{a}_{p_1+1} t^{\alpha_2} + \cdots)$$

must be in

$$f^{-(b-1)}\ k[t_1, t_1^{-1}, \cdots, t_{n-1}, t_{n-1}^{-1}, x_1, x_n, f_2^{-1}, \cdots, f_p^{-1}]$$

for every i. Since there is a non-zero a_l, and $\alpha_i \neq \alpha_j$ for $i \neq j$, $\sigma_{ij}(x, x_n)$ must be divided by f. Therefore we have

$$\sigma(x, x_n) = \tilde{\sigma}(x, x_n)\ diag\ (1, 1, \cdots, 1, \overset{l}{f}, 1, \cdots, 1)$$

where $\sigma(x, x_n)$ is in $GL_r(k[x_1, x_2, \cdots, x_n, f_2^{-1}, \cdots, f_p^{-1}])$ and $\det \sigma(x, x_n) = cf^{d-1}$. This contradicts the minimality of d. Hence $d = 0$. Therefore $\sigma(x, x_n)$ is not defined only at $f_2, \cdots, f_p$. This is a contradiction. So $\sigma(x, x_n)$ is in $GL_r(k[x, x_n])$.

By the same way we can take $\tau(x, x_n)$ in $GL_r(k[x, x_n])$ such that

$$\rho(1, t_n,\ ;\ x, x_n) = \tau(x, x_n)\ F\tau(x, t_n x_n)^{-1}$$

where F is a diagonal matrix whose diagonal elaments are of the form t^{β_i} (β_i : integers). We take $\tau(x, x_n)^{-1}\rho(t t_n\ ;\ x, x_n)\ \tau(tx, t_n x_n)$ in place of $\rho(t, t_n\ ;\ x, x_n)$. So we may assume that $\rho(1, t_n\ :\ x, x_n) = F$.

For this $\rho(t, t_n\ ;\ x, x_n)$ we take $\sigma(x, x_n)$ in $GL_r(k[x, x_n])$, by the above way, such that

$$\rho(t, 1\ ;\ x, x_n) = \sigma(x, x_n)\ E\sigma(tx, x_n)^{-1}.$$

Since

$$\rho(t, t_n ; 1, 1)=\rho(t, 1 ; 1, 1)\,\rho(1, t_n ; t, 1)$$
$$=\rho(1, t_n ; 1, 1)\,\rho(t, 1 ; 1, t_n)$$

we have

$$\sigma(1, 1)\, E\sigma(t, 1)^{-1} F = F\sigma(1, t_n)\, E\sigma(t, t_n).$$

Hence, by PROPOSITION 2.4, there is a scalar matrix H such that $H\sigma(t, t_n)^{-1}F^{-1}\sigma(t, 1)FH^{-1}$ is of the form

$$\begin{pmatrix} \boxed{*} & & \# \\ & \boxed{*} & \\ 0 & & \ddots \ \boxed{*} \end{pmatrix}$$

where elements in $*$ are in $k[t_n, t_n^{-1}]$ and elements in $\#$ are $b_{ij}(t_n)\,t^{\gamma_{ij}}$ in $k[t, t_n, t_n^{-1}]$ and γ_{ij} are positive integers determined by α_i. So we may assume that $B(t, t_n)=\sigma(t, t_n)^{-1}F^{-1}\sigma(t, 1)F$ is above from. Thus we have

$$\rho(t, t_n ; 1, 1)=\sigma(1, 1)\, EFB(t, t_n)^{-1}\sigma(t, t_n)^{-1}.$$

Hence

$$\rho(t, t_n ; x, x_n)=\rho(x, x_n ; 1, 1)^{-1}\rho(tx, t_nx_n ; 1, 1)$$
$$=\sigma(x, x_n)B(x, x_n)\, EFB(tx, t_nx_n)^{-1}\sigma(tx, t_nx_n)^{-1}$$

but this may not be defined at $x_i=0$ $(i=1, \cdots, n)$.

Since

$$B(0, x_n)\, EFB(0, t_nx_n)^{-1}=\sigma(0, x_n)^{-1}\rho(t, t_n ; 0, x_n)\sigma(0, t_nx_n)^{-1}$$

we can take a matrix D in $GL_r(k[x_n, x_n^{-1}])$ and $B(0, x_n)D$ is of the form

$$\begin{pmatrix} \boxed{*} & & \\ & \boxed{*} & 0 \\ 0 & & \ddots \ \boxed{*} \end{pmatrix}$$

by the same reason of PROPOSITION 2.3. Furthermore $B(x, x_n)D$ is same form of $B(x, x_n)$.

Since

$$(B(0, x_n)D)^{-1}(B(x, x_n)D)=\begin{pmatrix} 1 & & * \\ & 1 \ddots & \\ 0 & & 1 \end{pmatrix}$$

there is a matrix $B'(x, x_n)$ in $GL_r(k[x, x_n])$ such that

$$C(x, x_n)=B'(x, x_n)(B(0, x_n)D)^{-1}(B(x, x_n)D)$$
$$=\begin{pmatrix} 1 & & c_{ij}(x, x_n) \\ & 1 & \\ & & \ddots \\ 0 & & 1 \end{pmatrix}$$

where $c_{ij}(x, x_n)$ are 0 or in $k[x, x_n^{-1}]$ but in $k[x]$.

Since $C(x, x_n) EFC(tx, t_n x_n)^{-1}$ is in $GL_r(k[t, t^{-1}, t_n, t_n^{-1}, x, x_n])$, by seeing the term with respect to x_n, $C(x, x_n) EFC(tx, t_n x_n)^{-1}$ must be equal to EF. Then $\tilde{\sigma}(x, x_n) B'(x, x_n)^{-1} B(0, x_n) D$ satisfies THEOREM 2.2.

Q.E.D.

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