A note on the Jordan decomposition of derivations

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Abstract

As a linear map, a derivation of a K-algebra can be decomposed into semi-simple part and nilpotent part and the each part is also a K-derivation of the K-algebra. If the semi-simple part is not trivial the K-algebra is decomposed into eigenspaces.

§1 Jordan decompositions.

First, we recall the Jordan decomposition of an linear endomorphism of a finite dimensional vector space. Let K be an algebraically closed field and V be a finite dimensional vector space over K. If f is a K-linear endomorphism of V then there is a semi-simple K-linear endomorphism s and a nilpotent K-linear endomorphism n of V such that

$$f=s+n \text{ and } sn=ns.$$

Furthermore, n and s are uniquely determined by f. We denote s by f_s and n by f_n and we call f_s the semi-simple part of f and f_n the nilpotent part of f. Let $\lambda_1, \dots, \lambda_n$ be the eigen values of f_s and let $V_i=\{x \in V \mid f_s(x)=\lambda_i x\}$ for $i=1, 2, \dots, n$. Then V_i consists of elements x such that $(f-\lambda_i)^m(x)=0$ for some $m>0$. V is a direct sum of $V_1, \dots, V_n$.

From now on, we assume that $ch(K)=0$. We apply the Jordan decomposition to a finite dimensional K-algebra.

Theorem 1. Let A be a finite dimensional K-algebra and D a K-derivation of A. Then D_s and D_n are K-derivations of A.

Proof. Put $S=D_s$ and $N=D_n$. If $S=0$ there is nothing to prove. Let $S\neq 0$. Then A is a direct sum of eigenspaces A_λ of S. Let $x\in A_\lambda$ and $y\in A_\mu$. Then $S(x)=\lambda x$ and $S(y)=\mu y$. So, there are integers $n, m>0$ such that $(D-\lambda)^n(x)=0$ and $(D-\mu)^m(y)=0$. By the formula

$$(D-\lambda-\mu)^k(xy)=\sum_{i=0}^{k}\binom{k}{i}(D-\lambda)^i(x)\,(D-\mu)^{k-i}(y)$$

we get $(D-\lambda-\mu)^k(xy)=0$ for $k>n+m$. This shows that $xy\in A_{\lambda+\mu}$. Then,

$$S(xy)=(\lambda+\mu)xy=(\lambda x)y+x(\mu y)=S(x)y+xS(y).$$

Hence S is a K-derivation of A.

Let (A, m, K) be a complete local ring and let $\mathrm{Der}(A)$ be the set of K-derivations of A. It is well-known that $\mathrm{Der}(A)$ is an A-module and has a structure of a Lie algebra over K. We refer the reader to H. Matsumura's book for the fundamental properties of complete local rings and derivations.

In the suquel R denotes the formal power series ring $K[[X_1, \dots, X_n]]$ in $X_1, \dots, X_n$ over K and $A=R/I$ for some ideal I of R. Let ∂_i be the K-derivation of R given by $\partial_i(X_j)=\delta_{ij}$. For any K-derivation ∂ of A there exist $A_1, \dots, A_n \in R$ such that

$$\partial(X \bmod I)=A_1\partial_1(X)+\cdots+A_n\partial(X) \bmod I$$

for $X\in R$ and ∂ is denoted by $a_1\partial_1+\cdots+a_n\partial_n$, where $a_i=A_i \bmod I$.

Suppose that $a_i\in m$ for $i=1, \dots, n$. Then $\partial(m^k)\subseteq m^k$ by Leibniz rule. Hece, ∂ induces a K-derivation ∂_k of A/m^k such that

$$\pi_k\partial=\partial_k\pi_k,$$

where π_k: $A\rightarrow A/m^k$ is the natural projection.

Applying Theorem 1 to ∂_k, we have the Jordan decomposition

$$\partial_k=\partial_{ks}+\partial_{kn}.$$

There is a polynomial $f(X)\in K[X]$ such that $\partial_{ks}=f(\partial_k)$ and $f(0)=0$. Since ∂_k maps the maximal ideal of A/m^k into itself the same is true for ∂_{ks} and ∂_{kn}. Let ϕ_k: $A/m^{k+1}\rightarrow A/m^k$ be the natural projection. Then we have $\phi_k\partial_{k+1s}(m^k/m^{k+1})=0$ and we have a K-derivation S of A/m^k such that

$$\phi_k\partial_{k+1s}=S\phi_k.$$

Similarly, we have a K-derivation N of A/m^k such that

$$\phi_k\partial_{k+1n}=N\phi_k.$$

Let $\lambda_1, \dots, \lambda_r$ be eigenvalues of ∂_{k+1s} and decompose

$$A/m^{k+1}=A_1\oplus\cdots\oplus A_r,$$

where $A_i=\{x\in A/m^{k+1} \mid \partial_{k+1s}(x)=\lambda_i x\}$.
Then

$$A/m^k=\sum_{1\leqq i\leqq r}\phi_k(A_i)$$

For $a\in A_i$ we have

$$S(\phi_k(a))=\phi_k(\partial_{k+is}(a))=\lambda_i\phi_k(a).$$

Hence S is semi-simple and $\phi_k(A_i)$ is an eigenspace of S.

If $(\partial_{k+1n})^m=0$ we have

$$0=\phi_k(\partial_{k+1n})^m=N^m\phi_k.$$

Since ϕ_k is onto we have $N^m=0$.
From $\partial_{k+1n}\partial_{k+1s}=\partial_{k+1s}\partial_{k+1n}$ we have

$$NS\phi_k=N\phi_k\partial_{k+1s}=\phi_k\partial_{k+1n}\partial_{k+1s}=\phi_k\partial_{k+1s}\partial_{k+1n}=SN\phi_k.$$

Since ϕ_k is surjective we have $SN=NS$. Therefore, by the uniqueness of the Jordan decomposition, $\partial_{ks}=S$ and $\partial_{kn}=N$. Now we have

$$\phi_k\partial_{k+1s}=\partial_{ks}\phi_k \text{ and } \phi_k\partial_{k+1n}=\partial_{kn}\phi_k.$$

For $a\in A$, we put $a_k=\partial_{ks}(a \bmod m^k)$ and $b_k=\partial_{kn}(a \bmod m^k)$. Then we have

$$\phi_k(a_{k+1})=a_k \text{ and } \phi_k(b_{k+1})=b_k.$$

Since A is complete

$$(0,a_1, a_2, \dots, a_k, \dots)\in \varprojlim_k A/m^k=A \text{ and}$$
$$(0, b_1, b_2, \dots, b_k, \dots)\in \varprojlim_k A/m^k.$$

we can define two maps ∂_s and ∂_n, from A into itself, by

$$\partial_s(a)=(0, a_1, a_2, \dots, a_k, \dots) \text{ and}$$
$$\partial_n(a)=(0, b_1, b_2, \dots, b_k, \dots).$$

Clearly, ∂_s and ∂_n are K-derivations of A and $\partial=\partial_s+\partial_n$. Thus, we have proved:

Theorem 2. If ∂ is a K-derivation of a complete local ring (A, m, K) such that $\partial(m)\subseteq m$, then there are K-derivations ∂_s and ∂_n of A such that $\partial=\partial_n+\partial_n$ and, for $k=1, 2, \dots$, the K-derivations ∂_{ks} and ∂_{kn} of A/m^k induced from ∂_s and ∂_n are semi-simple and nilpotent, respectively, and $\partial_s\partial_n=\partial_n\partial_s$.

§2 Decomposition of complete local rings.

Let ∂ be a K-derivation of a complete local ring (A, m, K) such that $\partial(m)\subseteq m$ and $\partial_{2s}\neq 0$. If $\lambda_0=0, \lambda_1, \dots, \lambda_r$ are eigenvalues of ∂_{2s}, we can decompose

$$A/m^2=A_0\oplus A_1\oplus\cdots\oplus A_r,$$

where $A_i=\{x\in m/m^2|\partial_{2s}(x)=\lambda_i x\}$.
Since $\partial_{2s}(K)=0$ and $\partial_{2s}(m/m^2)\in m/m^2$, we have $A_i\in m/m^2$ if $\lambda_i\neq 0$. Fix λ_i $(\lambda_i\neq 0)$. Let $A_{ki}=\{x\in A/m^k|\partial_{ks}(x)=\lambda_i x\}$. As we have seen in §1, ϕ_k maps A_{k+1i} onto A_{ki}. For any $x\in A_i$, we can find $a_k\in A_{ki}$ $(k=2, 3, \dots)$ such that $a_z=x$, $a_k=\phi_k(a_{k+1})$ for $K=2, 3, \dots$. Then

$$a=(0, a_2, a_3, \dots)\in \varprojlim_k A/m^k=A,$$

and $\partial_s(a)=\lambda_1 a$.

Applying this argument to a basis of m/m^2 we can find minimal generators $x_1, x_2, \ldots x_n$ of m such that $\partial_s(x_i)=\lambda_{\sigma(i)}x_i$, for some eigenvalue $\lambda_{\sigma(i)}$ of ∂_{2s}. Let $\Lambda\subset K$ be an additive semi-group generated by $\lambda_0, \ldots, \lambda_r$. If x, $y\in A/m^k$ satisfy $\partial_{ks}(x)=\lambda x$, $\partial_{ks}(y)=\mu y$ and $xy\neq 0$, we have $\partial_{ks}(xy)=(\lambda+\mu)xy$ by the proof of Theoreml. Let $A_\lambda=\varprojlim_k A_{k\lambda}$, where $\lambda\in A$ and $A_{k\lambda}=\{x\in A/m^k|\partial_{ks}(x)=\lambda x\}$. Then

$$A_\lambda=\{a\in A|\partial_s(a)=\lambda a\} \text{ and } A_\lambda A_\mu\subseteq A_{\lambda+\mu}.$$

Now we can decompose

$$(1)\quad A=\sum_{\lambda\in\Lambda}A_\lambda.$$

Note that Λ is a countable set and let $\Lambda=\{\mu_1, \mu_2, \ldots\ldots\}$. (1) means that any $a\in A$ can be written uniquely in the form

$$a=a_1+a_2+\cdots+a_i+\cdots,$$

where $\partial_s(a_i)=\mu_i a_i$, and this infinite sum makes sense in the m-adic topology. We have proved:

Theorem 3. Let ∂ be a K-derivation of A such that $\partial(m)\subseteq m$ and $\partial_{2s}\neq 0$. Then, there is a countable semi-group $\Lambda=\{\mu_1, \mu_2, \ldots.\}$ contained in K such that any $a\in A$ can be written uniquely in the form

$$a=a_1+a_2+\cdots a_i+\cdots,$$

where a_i satisfies $\partial_s(a_i)=\mu_i a_i$ with $\mu_i\in\Lambda$. Moreover, if we put $A_\lambda=\{a\in A|\partial_s(a)=\lambda a\}$ for $\lambda\in\Lambda$, we have $A_\lambda A_\mu\subseteq A_{\lambda+\mu}$.

Example 1. Let I be an ideal of R generated by homogeneous polynomials.

Then, $\partial=x_1\partial_1+x_2\partial_2+\cdots+x_n\partial_n$, $x_i=X_i$ mod I, is a K-derivation of A. Clearly, ∂ is semi-simple and we have a decomposition

$$A=\sum_{\lambda\in\Lambda}A_\lambda, \text{ where } \Lambda=Z_{\geqq 0}.$$

A_n is the set of homogeneous elements of degree n.

Example 2. Let I be an ideal of R generated by monomials. Then, $x_i\partial_i$ is a K-derivation of A. For any elements $\alpha_1,\alpha_2, \ldots, \alpha_n\in K$, $\partial=\alpha_1, x_1\partial_1+\alpha_2x_2\partial_2+\cdots+\alpha_nx_n\partial_n$ is a semi-simple K-derivation of A. We have

$$A=\sum_{\lambda\in\Lambda}A_\lambda$$

, where $\Lambda=\{k_1\alpha_1+k_2\alpha_2+\cdots+k_n\alpha_n|k_i\in Z_{\geqq 0}\}$.

Example 3. Let $A=K[[X,Y,Z]]/(X^2+Y^2+Z^2+XYZ)$. Then,

$$(X+\frac{1}{2}YZ)\frac{\partial}{\partial Y}-(Y+\frac{1}{2}XZ)\frac{\partial}{\partial X}$$

induces a K-derivation ∂ of A. ∂_2 is given by $\partial_2(x)=-y$, $\partial_2(y)=x$, where x and y are the images of X and Y in A/m^2, respectively. The eigenvalues of ∂_2 are 0, i and $-i$, where $i^2=-1$. We have

$$A=\sum_{\lambda\in\Lambda}A_\lambda,$$

where $\Lambda=\{ki|k\in Z\}$.

§3 Exponential of derivation.

In this section, we assume that $K=\mathbb{C}$, the complex number field. For any $n\times n$ complex matrix A we can define the exponential

$$\begin{aligned}\exp(A)&=\lim_{n\to\infty}(E+A/n)^n\\&=\sum_{k=0}^{\infty}\frac{1}{k!}A^k\end{aligned}$$

of A, see Chapter 1 of [C] for details. If $AB=BA$ we have $\exp(A+B)=\exp(A)\exp(B)$. We consider the exponential of a $\mathbb{C}$-derivation of a finite dimensional $\mathbb{C}$-algebra.

Let A be a finite dimensional $\mathbb{C}$-algebra and ∂ be a $\mathbb{C}$-derivation of A. Fix a basis x_1, x_2, $\cdots$, x_n of A. Then ∂ is given by a matrix $D=(a_{ij})$. We define the exponential $\exp(\partial)$ to be the linear map given by $\exp(D)$. The following Lemma 4 and Corollary 5 are easy exercises, cf. Chapter 2 of [V]. Let G be the $\mathbb{C}$-algebra automorphism group of A.

Lemma 4. Let A be a finite dimensional $\mathbb{C}$-algebra. Then a $\mathbb{C}$-linear endomorphism ∂ of A is a $\mathbb{C}$-derivation of A if and only if for any $t\in\mathbb{R}$, $\exp(t\partial)$ is a $\mathbb{C}$-algebra automorphism of A.

Corollary 5. The Lie algebra of G is isomorphic to the space of $\mathbb{C}$-derivations of A.

We apply these results to a complete local $\mathbb{C}$-algebra. Let $(A, m, \mathbb{C})$ be a complete local ring of the form $\mathbb{C}[[X_1, X_2, \cdots X_n]]/I$. G denotes the $\mathbb{C}$-algebra automorphism group of A and, for $k=1, 2, \cdots$, G_k the $\mathbb{C}$-algebra automorphism group of A/m^k. Note that $G=\varprojlim_k G_k$. Let $\mathrm{Der}(A)$ be the set of $\mathbb{C}$-derivations of A and we put $D_k=\mathrm{Der}(A/m^k)$.

Definition 6. (1) An ideal I of A is called a D-ideal if $\partial(I)\subseteq I$ for all $\partial\in\mathrm{Der}(A)$.
(2) An ideal I of A is called a G-ideal if $\sigma(I)\subseteq I$ all $\sigma\in G$. If m is a D-ideal, we have $\mathrm{Der}(A)=\varprojlim_k D_k$. Let e_k be the exponential map $D_k\to G_k$ given by $e_k(\partial_k)=\exp(\partial_k)$ for $\partial_k\in D_k$. Let us assume that m is a D-ideal and let $\partial\in\mathrm{Der}(A)$. For $k=1, 2, \cdots$, ∂ induces a $\mathbb{C}$-derivation ∂_k of A/m^k. Then,

$$\sigma = (e_1(\partial_1), e_2(\partial_2), e_3(\partial_3), \cdots) \in \varprojlim_k G_k = G.$$

Hence, we can define a map Exp: $\mathrm{Der}(A) \to G$. If $\partial, \delta \in \mathrm{Der}(A)$ satisfy $\partial\delta = \delta\partial$ then $\mathrm{Exp}(\partial + \delta) = \mathrm{Exp}(\partial)\mathrm{Exp}(\delta)$. Let $\partial = \partial_s + \partial_n$ be the Jordan decompostion of ∂. Then we have $\mathrm{Exp}(\partial) = \mathrm{Exp}(\partial_s)\mathrm{Exp}(\partial_n)$.

Theorem 7. If m is a D-ideal then a G-ideal is a D-ideal.

Proof. Let I be a G-ideal and $\partial \in \mathrm{Der}(A)$. Put $\sigma_t = \mathrm{Exp}(t\partial)$, $t \in \mathbb{R}$. Then, for $k = 1, 2, \cdots$, we have an automorphism $\sigma_{kt} = \exp(t\partial_k)$ of A/m^k. For any $x \in I + m^k/m^k$ we have $\partial_k(x) = \lim_{t \to 0} t^{-1}(\sigma_{kt}(x) - x) \in I + m^k/m^k$ because I is a G-ideal and A/m^k is a finite dimensional $\mathbb{C}$-algebra. This implies that $\partial(I) \subseteq I + m^k$ for $k = 1, 2, \cdots$. It follows that

$$\partial(I) \subseteq \bigcap_{k \geq 0} (I + m^k) = I.$$

It is natural to conjecture that Theorem 7 is true for any base field of characteristic 0. If $K \neq \mathbb{C}$ then $\mathrm{Exp}(\partial_s)$ does not make sense in general but $\mathrm{Exp}(\partial_n)$ does, bacause ∂_{kn} is nilpotent for all k.

We collect several results on D-deals.

Lemma 8. Let K be a field with $\mathrm{ch}(K) = 0$ and (A, m, K) a complete local K-algebra. Then:

(1) If I and J are D-ideals of A then $\sqrt{I}$, $I \cap J$, $I + J$, IJ and I: J are D-ideals.
(2) If I is a D-ideal then associated primes are D-ideals.
(3) A is regular if and only if 0 and A are the only D-ideals of A.

Proof. See [S1] and [S2].

Proposition 9. If A is regular then non-zero G-ideals are m-primary.

Proof. Let I a G-ideal which is not m-primary. We may assume that I is a radical ideal which is not maximal. Let $a \in m^2$ be an elment which is not a zero-divisor of A/I and ∂ be a C-derivation of A. Then $a\partial$ induces a nilpotent derivation of A/m^k, for $k = 1, 2, \cdots$. Hence $\mathrm{Exp}(a\partial)$ is a well defined automorphim of A. For any $x \in I$, we have $\mathrm{Exp}(a\partial)(x) - x \in aA \cap I = aI$ and $a^{-1}(\mathrm{Exp}(a\partial)(x) - x) \in I$. But, we have $a^{-1}(\mathrm{Exp}(a\partial)(x) - x) = \partial(x) + ab$ for some $b \in A$. Therefore, we have $\partial(I) \subseteq I + aA$. Replacing a by a^k, we have $\partial(I) \subseteq I + a^k A$ for $k = 1, 2, \cdots$. Hence we have $\partial(I) \subseteq I$, I is a D-ideal and, by Lemma 8, $I = 0$.

Theorem 10. Let $(A, m, \mathbb{C})$ be a complete local $\mathbb{C}$-algebra. If A is not regular and m in a D-ideal then, for $p \in \mathrm{Spec}A$, A_p is not regular if p containes a G-ideal I such that $\mathrm{ht}(I) > 0$.

Proof. It is enough to show that $V(I)$ is contained the singular locus $\mathrm{Sing}(A)$ of A. Let J be the defining ideal of $\mathrm{Sing}(A)$. Suppose that there is a minimal prime p of I such that $p \not\supseteq J$. By Lemma 8 and Theorem 6, p is a D-ideal. We can find a system of parameters x_1, $x_2, \cdots, x_d$ of A so that $x_1, x_2, \cdots, x_r$ form a system of parameters of A/p. Then, $\mathbb{C}[[x_1, x_2, \cdots, x_r]] \cap p = 0$ and A/p is finte over the image of $\mathbb{C}[[x_1, x_2, \cdots, x_r]]$ in A/p. Let $S = \mathbb{C}[[x_1, x_2, \cdots, x_r]]$ and L the field of fractions of S. Then the field of fractions M of A/p is a finite separable extension of L. We apply the theory of universally finite module of differentials to our complete local rings. See [SS] for fundamental theory of universally finite module of differentials, see also [K]. Let Ω_A be the universally finite module of differentials of A over $\mathbb{C}$. Then $\mathrm{Der}(A)$ is isomorphic to $\mathrm{Hom}_A(\Omega_A, A)$. Since Ω_A is a finite A-module and A_p is flat over A, we have $\mathrm{Der}(A_p) = \mathrm{Hom}_A(\Omega_A \otimes_A A_p, A_p) = \mathrm{Der}(A) \otimes_A A_p$. Then we have $\partial_p(A_p) \subseteq pA_p$ for all $\partial \in \mathrm{Der}(A_p)$. Let B be the pA_p-adic conpletion of A_p. By [SS], the universally finite module of differentials of B over $\mathbb{C}$ is isomorphic to $\Omega_A \otimes_A B$ and hence, we have $\mathrm{Der}(B) = \mathrm{Der}(A_p) \otimes_A B$. From the exact sequence

$$\Omega_S \otimes_A A \to \Omega_A \to \Omega_{A/S} \to 0,$$

we have $\mathrm{Der}_L(B) = \mathrm{Hom}_A(\Omega_{A/S}, B) \subset \mathrm{Der}(B)$. Since M is finite separable over L we have $\mathrm{Der}_M(B) = \mathrm{Der}_L(B)$. We see that $\mathrm{Der}_M(B)$ is a submodule of $\mathrm{Der}(B) = \mathrm{Der}(A) \otimes_A B$. Hence, pB is a D-ideal of $B = M[[Y_1, Y_2, \cdots, Y_r]]$. By Lemma 8, we get $pB = 0$ and consequenly, $pA_p = 0$, but this contradicts to $\mathrm{ht}(p) > 0$. Therefore, every minimal prime of I containes the defining ideal of $\mathrm{Sing}(A)$.

References

[C] C. Chevalley, Theory of Lie Groups, Princeton University Press, Princeton, 1946.

[K] E. Kunz, Kähler Differentials, Vieweg, Braunschweig, 1986.

[M] H. Matsumura, Commutative Ring Theory, Cambridge University Press, Cambridge, England, 1986.

[S1] A. Seidenberg, Derivations and Integral Closure. Pac. J. Math. 16, 167-173 (1966).

[S2] A. Seidenberg, Differential Ideals in Rings of Finitely Generated Type. Amer. J. Math. 89, 22-42 (1967).

[SS] G. Scheja and U. Storch, Differentielle Eigenschaften der Lokalisierungen analytischer Algebren. Math. Ann. 197, 137-170 (1970).