The field of constants for a linear derivation of a rational function field

Norio Yamauchi

Abstract

Let k be a field of characteristic 0 and $K = k(X_1, \ldots, K_n)$ a rational function field over k in n variables. In this paper we study the field K^δ of constants for a linear k-derivation δ of K. The main result is THEOREM 4.3, which gives an explicit description of generetons of K^δ over k when k is algebraically closed.

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1 Jordan decomposition

Let k be a field and $K = k(X_1, \ldots, X_n)$ a rational function field over k. We set $D_0 = \{\delta \in \mathrm{Der}_k(K) \mid \delta(X_i)$ is a linear form in $X_1. \ldots, X_n$ for $i = 1, \ldots, n\}$ and $V = kX_1 + \cdots + kX_n (\subset K)$. There exists an isomorphism $D_0 \to \mathrm{End}_k(V)$ of k-vector spaces given by $\delta \mapsto \varphi_\delta =$ the restriction of δ to V. Its inverse is given by $\varphi \mapsto \delta_\varphi = \Sigma_{i=1}^n \varphi(X_i)\partial/\partial X_i$ for $\varphi \in \mathrm{End}_k(V)$.

Let $\delta \in D_0$. We say δ is semisimple (or diagonal or nilpotent) if φ_δ has the same property (as a k-endmorphism). Using the Jordan decomposition of φ_δ and the isomorphisms above, we see that there exist $\delta_s, \delta_n \in D_0$ satisfying the conditions : $\delta = \delta_s + \delta_n$, δ_s is semisimple and δ_n is nilpotent. Thus we have the following, which shows that when k is algebraically closed, any liner k-derivation can be expressed in the 'canonical' form :

LEMMA 1.1 *With the same notation as above, assume that* k contains all the roots of the characteristic polynomial of φ_δ. *Then we can chose a set* $\{X_{01}, \ldots, X_{0s}, X_{10}, \ldots, X_{1l_1}, \ldots, X_{t0}, \ldots, X_{tl_t}\}$ *of generators of* K *over* k *such that*

$\delta(X_{0i}) = \lambda_i X_{0i}$ $(i = 1, \ldots, s)$,

$\delta(X_{i0}) = \mu_i X_{i0}$, $\delta(X_{ij}) = \mu_i X_{ij} + X_{ij-1}$, $(i = 1, \ldots, t, j = 1, \ldots, l_i)$.

Throughout the rest of this paper, k denotes a field of characteristic 0. For a field F and a derivation Δ of F we denote by F^{Δ} the field of constants for Δ, that is, $F^{\Delta} = \{f \in F | \Delta(f) = 0\}$.

2 Diagonal case

In this section we alway assume that $K = k(X_1, \ldots, X_n)$ is a rational function field over k and that δ is a k-derivation of K such that $\delta(X_i) = \lambda_i X_i$ for some $\lambda_i \in k$ $(1 \leq i \leq n)$.

First we recall the notion of the multidegree of a polynomial. Let $f = \Sigma_{\alpha \in N^n} a_\alpha X^\alpha \in k[X_1, \ldots, X_n]$ be a nonzero polynomial. The term $a^\alpha X^\alpha$ is called the homogeneous term of f of multidegree α and is denoted by f_α. Let $\prec$ denote the lexicographic order of N^n. The multidegree m-deg(f) of f (with respect to $\prec$) is $\max\{\alpha \in N^n | a_\alpha \neq 0\}$, where max is taken with respect to $\prec$. When m-deg$(f) = \alpha$, f_α is called the leading term of f. We note that m-deg$(\delta(f)) \leq$ m-deg(f) because $\delta(X^\alpha) = \Sigma_{i=1}^n \alpha_i \lambda_i X^\alpha$ for a monomial $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

Sinece δ is a k-endomorpophism of K, we can consider eigenvalues, eigenvectors and eigenspaces of δ. We write $V(\lambda)$ for the eigenspace of δ, belonging to λ. We note that $V(0) = K^\delta$ and that $V(\lambda)$ is not only a k-vector space but also a K^δ-vector space. If $f, g \in V(\lambda)$ and $g \neq 0$ then $f/g \in K^\delta$. Hence $\dim_{K^\delta} V(\lambda) = 1$ if $V(\lambda) \neq 0$.

For a vector $m = (m_1, \ldots, m_m) \in Z^n$, we have $\delta(X^m) = \Sigma_{i=1}^n m_i \lambda_i X^m$ and hence X^m is an eigenvector of δ and $\Sigma_{i=1}^n m_i \lambda_i$ is an eigenvalue of δ.

LEMMA 2.1 *Let $f, g \in k[X_1, \ldots, X_n]$ be nonzero polynomials such that $f/g \in X(\lambda)$ for some $\lambda \in k$. Let $f_\alpha = aX^\alpha$ and $g_\beta = bX^\beta$ be the leading terms of f and g, respectively. Then we have* $\Sigma_{i=1}^n (\alpha_i - \beta_i)\lambda_i = \lambda$ and $f_\alpha / g_\beta \in V(\lambda)$.

Proof. By the assumption we have $\delta(f)g - f\delta(g) = \lambda fg$. It is immediate to see that $(\delta(f)g - f\delta(g))_{\alpha+\beta} = ab\Sigma_{i=1}^n (\alpha_i - \beta_i)\lambda_i X^{\alpha+\beta}$ and that $(\lambda fg)_{\alpha+\beta} = ab\lambda X^{\alpha+\beta}$. Hence we have $\Sigma_{i=1}^n (\alpha_i - \beta_i)\lambda_i = \lambda$ and $f_\alpha / g_\beta = (a/b) X^{\alpha-\beta} \in V(\lambda)$. ∎

COROLLARY 2.2 *If $\lambda_1, \ldots, \lambda_n$ are linearly independent over Q then $K^\delta = k$.*

Proof. We use the same notation as in the lemma with $\lambda = 0$. By the lemma $\Sigma_{i=1}^n (\alpha_i - \beta_i)\lambda_i = 0$. Hence m-deg$(f)$ = m-deg(g) by the assumption that $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\boldsymbol{Q}$. Since $(bf - ag)/g = b(f/g) - a \in K^{\delta}$ we have m-deg $(bf - ag) =$ m-deg(g) if $bf - ag \neq 0$. But it is obvious that if $bf - af \neq 0$ then m-deg $(bf - ag) <$ m-deg(f). Hence we have $bf - af = 0, f/g = a/b \in k$. ■

COROLLARY 2.3 *Let $\lambda = \Sigma_{i=1}^{n} m_i\lambda_i$ be an eigenvalue of δ with $m = (m_1, \ldots, m_n) \in \boldsymbol{Z}^n$. Then $V(\lambda) = K^{\delta}X^m$.*

The following lemma is well known (see *e. g.* Kunz[1] §5 Exercise 1)).

LEMMA 2.4 *Let $v_1, \ldots, v_n$ be vectors in $\boldsymbol{Z}^n$. Then the monomials $X^{v_1}, \ldots, X^{v_n}$ are algebraically independent over k if and only if $v_1, \ldots, v_n$ are linearly independent over $\boldsymbol{Z}$.*

Now we can prove the following, which gives generators of the field of constans for a diagonal derivation :

PROPOSITION 2.5 *Let $N = \{(m_1, \ldots, m_m) \in \boldsymbol{Z}^n \mid \Sigma_{i=1}^{n} m_i\lambda_i = 0\}$. Then $K^{\delta} = k(\{X^m \mid m \in N\})$.*

Proof. We put $K_0 = k(\{X^m \mid m \in N\})$. It is obvious that $K_0 \subset K^{\delta}$. We show the opposite inclusion. Since N is a submodule of $\boldsymbol{Z}^n$ we can take a basis B_0 of N. We set $M_0 = \{X^m \mid m \in B_0\}$. It is easy to see that $K_0 = k(M_0)$. Let $E = \{e_1, \ldots, e_n\}$ be the standard basis of $\boldsymbol{Z}^n$. Then there exists a subset B_1 of E such that B_1 has $(n - r)$ elements ($r =$ rank(N) and $B = B_0 \cup B_1$ is linearly independent over $\boldsymbol{Z}$. We may assume $B_1 = \{e_{r+1}, \ldots, e_n\}$. We set $M_1 = \{X^{\alpha} \mid \alpha \in B\}$ and $K_1 = k(M_1) = K_0(X_{r+1}, \ldots, X_n)$. Then, by LEMMA 2.4, K_1 is a purely transcendental extension of K_0. The restriction ∂ of δ to K_1 is a K_0-derivation of K_1 such that $\partial(X_{r+1}) = \lambda_{r+1}X_{r+1}, \ldots, \partial(X_n) = \lambda_n X_n$. Since $\lambda_{r+1}, \ldots, \lambda_n$ are linearly independent over $\boldsymbol{Q}$ by the choice of B_1 we have $K_1^{\partial} = K_0$ by COROLLARY 2.2.

On the other hand, since tr. $\deg_k K_1 = n$, by LEMMA 2.4, $K = K_1(X_1, \ldots, X_r)$ is an algebraic extension of K_1. Hence there exists a subset I of $\boldsymbol{N}^r$ such that $M = \{M_{\alpha} = X_1^{\alpha_1} \cdots X_r^{\alpha_r} \mid \alpha = (\alpha_1, \ldots, \alpha_r) \in I\}$ is a basis of K as a K_1-vector space. Note that if $\alpha \in I$ and $\alpha \neq 0$ then $\Sigma_{i=1}^{n} \alpha_i\lambda_i \neq 0$ for any $\alpha_{r+1}, \ldots, \alpha_n \in \boldsymbol{Z}$. Now let us take an element $f = \Sigma_{\alpha \in I} f_{\alpha} M_{\alpha}$ of K^{δ} $(f_{\alpha} \in K_1)$. Then $\delta(f) = \Sigma_{\alpha \in I}(\delta(f_{\alpha}) + (\Sigma_{i=1}^{r}\alpha_i\lambda_i)f_{\alpha})$

M_α and hence $\delta(f_\alpha) + (\Sigma_{i=1}^r \alpha_i\lambda_i)f_\alpha = 0$. By LEMMA 2.1 and COROLLARY 2.3 it follows that if $f_\alpha \neq 0$ then there exist $\lambda_{r+1}, \ldots, \lambda_n \in \boldsymbol{Z}$ and $c_\alpha \in K_0$ such that $f_\alpha = c_\alpha X_{r+1}^{\alpha_{r+1}} \cdots X_n^{\alpha_n}$ and $\Sigma_{i=r+1}^n \alpha_i\lambda_i = -\Sigma_{i=1}^r \alpha_i\lambda_i$. Hence we have $f_\alpha = 0$ for $\alpha \neq 0$ and $f_0 \in K_0$. This shows that $K^\delta \subset K_0$. ∎

3 Nilpotent case

In this section we concern with the field of constants for a nilpotent derivation. Let $R = k[Y_1, \ldots, Y_n]$ be a polynomial ring over k in n variables. Let ∂ be a k-derivation such that $\partial(Y_1) = 1$, $\partial(Y_2) = Y_1, \ldots, \partial(Y_n) = Y_{n-1}$. We set

$$f_i = \sum_{k=0}^{i-2} \frac{(-1)^k}{k!} Y_1^k Y_{i-k} + \frac{(-1)^{i-1}(i-1)}{i!} Y_1^i \quad (i = 2, \ldots, n).$$

It is known that the ring R^∂ of constants for δ is generated by $\{f_2, \ldots, f_n\}$ over k (Onoda [2] Lemma 1). A similar result holds for the field of constants. For completeness we state it with a proof.

LEMMA 3.1 *Let the notation be as above. We write L for the quotient field of R. Then $L^\partial = k(f_2, \ldots, f_n)$ and $L = L^\partial(Y_1)$.*

Proof. It is easy to see that $f_i \in L^\partial$ for $i = 2, \ldots, n$. Since $f_i = Y_i + g_i$, $g_i \in k[Y_1, \ldots, Y_{i-1}]$ $(i = 2, \ldots, n)$ we have $L = k(Y_1, f_2, \ldots, f_n)$. Moreover ∂ is a $k(f_2, \ldots, f_n)$-derivation and $\partial(Y_1) = 1$. Hence $L^\partial = k(f_2, \ldots, f_n)$. ∎

COROLLARY 3.2 *Let $K = k(X_0, X_1, \ldots, X_n)$ be a rational function field in $(n+1)$ variables over k. Let $\delta = \Sigma_{i=1}^n X_{i-1}\partial/\partial X_i$ be a k-derivation of K. We set*

$$f_1 = X_1/X_0$$

$$f_i = \sum_{k=0}^{i-2} \frac{(-1)^k X_1^k K_{i-k}}{k!} + \frac{(-1)^{i-1}(i-1)}{i!}\left(\frac{X_1}{X_0}\right)^i \quad (i = 2, \ldots, n).$$

Then we have $K = k(X_0, f_1, \ldots, f_n)$ and $K^\delta = k(X_0, f_2, \ldots, f_n)$.

Proof. We put $Y_i = X_i/X_0$ $(i = 1, \ldots, n)$ and $L = k(Y_1, \ldots, Y_n)$. Then $\delta(Y_1) = 1$, $\delta(Y_2) = Y_1, \ldots, \delta(Y_n) = Y_{n-1}$, $K = L(X_0)$. The restriction ∂ of δ to L is a k-deriation of L. Hence $L^\partial = k(f_2, \ldots, f_n)$ and $L = L^\partial(f_1)$ by LEMMA 3.1. Since δ is a $k(X_0, f_2, \ldots, f_n)$-derivation, $K = k(X_0, f_2, \ldots, f_n)(f_1)$ and $\delta(f_1) = 1$ we have $K^\delta = k(X_0, f_2, \ldots, f_n)$. ∎

PROPOSITION 3.3 *Let* $K = k(X_{10}, \ldots, X_{1l_1}, \ldots, X_{t0}, \ldots, X_{tl_t})$ *be a rational function field over* k. *Let* δ *be a* k*-derivation of* K *such that*

$$\delta(X_{i0}) = 0,\ \delta(X_{ij}) = X_{ij-1}\ (i = 1, \ldots, t,\ j = 1, \ldots, l_i).$$

We set

$$S_0 = \{X_{i0} \mid i = 1, \ldots, t\}$$

$$S_i = \{\sum_{k=0}^{j-2} \frac{(-1)^k}{k!} \frac{X_{i1}^k X_{ij-k}}{X_{i0}^{k+1}} + \frac{(-1)^{j-1}(i-1)}{j!}\left(\frac{X_{i1}}{X_{i0}}\right) \mid j = 2, \ldots, l_i\}\ (i = 1, \ldots, t),$$

$$S_{t+1} = \{X_{i1}/X_{i0} - X_{i+11}/X_{i+10} \mid i = 1, \ldots, t-1\} \text{ and}$$

$$S = \bigcup_{i=0}^{t+1} S_i.$$

Then we have $K^\delta = K(S)$.

Proof. We put $K_0 = k(S)$. It is obvious that $\delta = 0$ on S_0 and S_{t+1}. Moreover $\delta = 0$ on S_i for $i = 1, \ldots, t$ by COROLLARY 3.2. Hence δ is a K_0-derivation. Since $K = K_0(X_{11}/X_{10})$ and $\delta(X_{11}/X_{10}) = 1$ we have $K^\delta = K_0$. ■

4 Main result

In this section we give a set of generetors of the field of constants for a linear k-derivation when k is algebraically closed. First of all we recall the following

LEMMA 4.1 (SUZUKI [3] LEMMA 4) *Let* K *be a field of characteristic* 0, d *a derivation of* K. *Let* D *be an extension of* d *to a rational function field* $K(x)$ *in one variable such that* $D(x) \in K \setminus d(K)$. *Then* $K(x)^D = K^d$.

LEMMA4.2 *Let* $K = k(X_1, \ldots, X_n)$ *be a rational function field in* n *variables over* k *and* δ *a* k*-derivation of* K *suth that* $\delta(X_i) = \lambda_i X_i (\lambda_i \in k, 1 \leq i \leq n)$. *Let* $L = K(Y_1, \ldots, Y_m)$ *be a rational function field in* m *variables over* K *and* Δ *an extension of* δ *such that* $\Delta(Y_j) = 1$ $(j = 1, \ldots, m)$. *Then* $L^\Delta = k(\{X^m \mid m = (m_1, \ldots, m_n) \in \boldsymbol{Z}^n, \Sigma_{i=1}^n m_i\lambda_i = 0\} \cup \{Y_1 - Y_2, \ldots, Y_{m-1} - Y_m\}) = K^\delta(Y_1 - Y_2, \ldots, Y_{m-1} - Y_m)$.

Proof. We put $F_0 = k(Y_1 - Y_2, \ldots, Y_{m-1} - Y_m)$ and $F = F_0(X_1, \ldots, X_n)$. Then the restriction d of Δ to F is a F_0-derivation with $d(X_i) = \lambda_i X_i (1 \leq i \leq n)$. Hence, by PROPOSITION 2.5, $F^d = F_0\{X^m \mid m = (m_1, \ldots, m_m) \in \boldsymbol{Z}^n, \Sigma_{i=1}^n m_i\lambda_i = 0\}) = K^\delta(Y_1 - Y_2, \ldots, Y_{m-1} - Y_m)$. By LEMMA 4.1 it suffices to show that $\Delta(\varphi) \neq 1$ for any

$\varphi \in F$ because $L = F(Y_1)$ is a rational function field over F and $\Delta(Y_1) = 1$. Assume that there exists $\varphi = f/g$ satisfying $\Delta(\varphi) = 1$, where f and g are polynomials in $F_0[X_1, \ldots, X_n]$. Let aX^{α} and bX^{β} be the leading terms of f and g, respectively. Then we see that $\Delta(f)g - f\Delta(g) = ab\Sigma_{i=1}^{n}(\alpha_i - \beta_i)\lambda_i X^{\alpha+\beta}$ + (terms of multidegree $< \alpha + \beta$) and $g^2 = b^2X^{2\beta}$ + (terms of multidegree $< 2\beta$). Hence we have $\alpha = \beta$ and $a\ \Sigma_{i=1}^{n}(\alpha_i - \beta_i)\lambda_i = b \neq 0$, a contradiction. ∎

Now we are in a position to prove the following, which is our main result :

THEOREM 4.3 *Let k be an algebraically closed field of characteristic* 0 *and $K = k(X_1, \ldots, X_n)$ a rational function field over k. Let δ be a linear k-derivation of K. By* LEMMA 1.1 *we can take a set $\{X_{01}, \ldots, X_{0s}, X_{10}, \ldots, X_{1l_1}, \ldots, X_{t0}, \ldots, X_{tl_t}\}$ of generators of K over k such that*

$$\delta(X_{0i}) = \lambda_i X_{0i} \ (i = 1, \ldots, s),$$

$$\delta(X_{i0}) = \mu_i X_{i0},\ \delta(X_{ij}) = \mu_i X_{ij} + X_{ij-1},\ (i = 1, \ldots, t,\ j = 1, \ldots, l_i).$$

We set

$$S_0 = \{X_{01}^{m01} \cdots X_{0s}^{m0s} X_{10}^{m10} \cdots X_{t0}^{mt0} \mid m_{01}\lambda_1 + \cdots + m_{0s}\lambda_s + m_{10}\mu_1 + \cdots + m_{1t}\mu_t = 0\},$$

$$S_i = \{\sum_{k=0}^{j-2} \frac{(-1)^k}{k!} \frac{X_{i1}^k X_{ji-k}}{X_{i0}^{k+1}} + \frac{(-1)^{j-1}(j-1)}{j!}\left(\frac{X_{i1}}{X_{i0}}\right)^j \mid j = 2, \ldots, l_i\}\ (i = 1, \ldots, t),$$

$$S_{t+1} = \{X_{i1}/X_{i0} - X_{i+11}/X_{i+10} \mid i = 1, \ldots, t-1\}\ \textit{and}$$

$$S = \bigcup_{i=0}^{t+1} S_i.$$

Then we have $K^{\delta} = k(S)$.

Proof. We put $L = k(\cup_{i=1}^{t+1} S_i)$. Then we see that $\delta = 0$ on L and that K is a rational function field over L in variables $X_{01}, \ldots, X_{0s}, X_{10}, \ldots X_{to}, X_{11}/X_{10}$. Since $\delta(X_{0i}) = \lambda_i X_{0i}$ $(1 \leq i \leq s)$, $\delta(X_{i0}) = \mu_i X_{i0}$ $(1 \leq i \leq t)$ and $\delta(X_{11}/X_{10}) = 1$, by LEMMA 4.2, we have $K^{\delta} = L(S_0) = k(S)$. ∎

COROLLARY 4.4 *K^{δ} is a rational function field over k.*

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