Note on equivariant vector bundles on a ruled surface.

Tamafumi Kaneyama

§1. Introduction

Let k be an algebraically closed field of any characteristic. Let X be a complete non-singular variety over k. We say that X is an almost homogeneous variety if an algebraic torus T acts on X and X has a dense orbit. We may assume that $\dim(T)=\dim(X)$. The vector bundle E over an almost homogeneous variety X is equivariant by definition if there is an isomorphism

$$\phi_t : t^*E \tilde{\rightarrow} E$$

for any t in T where t^*E is the pull-back of E by the translation $t : X \rightarrow X$. In [2], we study equivariant vector bundles over an almost homogeneous variety. In this paper we intend to describe the concrete form of indecomposable equivariant vector bundles of rank 2 over a rational ruled surface $F_m = \mathbf{P}(O_{\mathbf{P}^1} \oplus O_{\mathbf{P}^1}(m))$ which is almost homogeneous under the standard torus action.

§2. Notations and preliminaries

Let Ξ be the character group $Hom_{k\text{-}gr}(T, G_m)$ of T where G_m is the multiplicative group. Let Γ be the one-parameter subgroups $Hom_{k\text{-}gr}(G_m, T)$ of T which is the dual $\boldsymbol{Z}$-module of Ξ by the pairing $\langle\ ,\ \rangle$. We extend the pairing to $\boldsymbol{Q}$ and denote

$$\langle \xi, \phi \rangle = \phi(\xi)$$

where $\xi \in \Xi_{\boldsymbol{Q}} = \Xi \otimes_{\boldsymbol{Z}} \boldsymbol{Q}$ and $\phi \in \Gamma_{\boldsymbol{Q}}$.

For the detailed description of almost homogeneous varieties, we refer to [3], [4].

Let X be an almost homogeneous variety defined by a cone complex $(\Gamma, \boldsymbol{C})$. We define the data (m, P) as follows:

(i) $m : Sk^1(\boldsymbol{C}) = \boldsymbol{C}^1 = \{\phi_1, \cdots, \phi_\alpha\} \rightarrow \mathbf{Z}^{\oplus r}$

sending ϕ to $m(\phi) = (m(\phi)_1, \cdots, m(\phi)_r)$ where $Sk^1(\boldsymbol{C})$ is a set of 1-dimensional cones of $\boldsymbol{C}$, and for every C in $\boldsymbol{C}^n$

$$m_C : C \cap Sk^1(\boldsymbol{C}) \rightarrow \boldsymbol{Z}^{\oplus r}$$

so that there is a permutation τ such that

$$m_C(\phi) = (m_C(\phi)_1, \cdots, m_C(\phi)_r) = (m(\phi)_{\tau(1)}, \cdots m(\phi)_{\tau(r)})$$

for every ϕ in $C \cap Sk^1(\boldsymbol{C})$.

or equivalently

(i′) $\xi : \boldsymbol{C}^n \rightarrow \Xi^{\oplus r}$

sending C to $\xi(C) = (\xi(C)_1, \cdots, \xi(C)_r)$ such that for every pair of cones C, C' in $\boldsymbol{C}^n$ there exists a permutation τ so that

$$\phi_j(\xi(C)_i) = \phi_j(\xi(C')_{\tau(i)})$$

for every i and every ϕ_j in $C \cap C'$.

(ii) $P : \boldsymbol{C}^n \times \boldsymbol{C}^n \to GL_r(k)$

sending (C, C') to $P(C, C') = (P(C, C')_{ij})$ such that $P(C, C')_{ij} \neq 0$ only if $m_{C'}(\phi)_i \geqq m_C(\phi)_j$ for every ϕ in $(C \cap C') \cap Sk^1(\boldsymbol{C})$ and such that

$$P(C, C')\, P(C', C'') = P(C, C'')$$

for every C, C', C'' in $\boldsymbol{C}^n$.

Two data (m, P) and (m', P') are equivalent if there exists a permutation $\sigma = \sigma(C)$ in $\mathfrak{B}_r$ such that

$$(m_C(\phi)_1, \cdots, m_C(\phi)_r) = (m'_C(\phi)_{\sigma(1)}, \cdots, m'_C(\phi)_{\sigma(r)})$$

for every C in $\boldsymbol{C}^n$ and ϕ in $C \cap Sk^1(\boldsymbol{C})$ and if there exists

$$\rho : \boldsymbol{C}^n \to GL_r(k)$$

such that

$$P'(C, C') = \rho(C)^{-1} P(C, C') \rho(C')$$

for every C and C' in $\boldsymbol{C}^n$

Then we have the following theorem in [2] on an equivariant vector bundle over an almost homogeneous variety. We say an equivariant vector bundle (E, ϕ_t) T-linearized if, for every pair of t, t' in T

$$\phi_{tt'} = \phi_{t'} \cdot t'^* \phi_t$$

holds.

THEOREM 2.1 *Let (X, T) be a complete non-singular almost homogeneous variety defined by a cone complex $(\Gamma, \boldsymbol{C})$. Then the set of T-linearized vector bundles of rank r up to T-isomorphism corresponds bijectively to the set of data (m, P) up to equivalence.*

§3. Line bundles on an almost homogeneous variety

Let (X, T) be a complete non-singular almost homogeneous variety defined by the cone complex $(\Gamma, \boldsymbol{C})$ unless stated. Let D_i be the divisor corresponding to ϕ_i in $Sk^1(\boldsymbol{C}) = \{\phi_1, \cdots, \phi_d\}$. Then $O_X(D_i)$ is defined by the data (m, P) where

$$m(\phi_i) = -1, \; m(\phi_j) = 0, \; (j \neq i)$$

and $P(C, C') = I$ for every pair of C, C' in $\boldsymbol{C}^n$. This section is mostly due to T. Oda.

PROPOSITION 3.1. *Let (X, T) be a complete almost homogeneous variety defined by a cone complex $(\Gamma, \boldsymbol{C})$. Then the following conditions are equivalent.*

(1) $D = \sum_{i=1}^{d} m_i D_i$ *is a Cartier divisor*

(2) D *is principal on each T-stable affine open subset.*

(3) *For every C in $\boldsymbol{C}$, there exists ξ_c such that*

$$\phi_i(\xi_c) = -m_i$$

for every ϕ_i in $C \cap Sk^1(\boldsymbol{C})$

proof. (2)⇨(1) is obvious.

(1)⇨(2). For C in $\boldsymbol{C}$, let $U_c = \mathrm{Spec}\,(A_c)$ be a T-stable affine open subset corresponding to C where $A_c = k[e(\Xi \cap C^*)]$.

Put

$$M_C(D) = \{\xi \in \Xi \mid \phi_i(\xi) \geqq -m_i \text{ for every } \phi_i \text{ in } C \cap Sk^1(\boldsymbol{C})\} .$$

Then

$$M_C = \Gamma(U_C,\ O_X(D)) = ke\ (\mu_C(D)).$$

Suppose D is a Cartier on U_c then

$$M_c \cdot (A_c : M_c) = A_c.$$

Therefore we have

$$\mu_C(D) + \{\eta \in \Xi \mid \eta + \mu_C(D) \subset \Xi \cap C^*\} = \Xi \cap C^*.$$

Since $\Xi \cap C^*$ contains zero, there exists ξ_c in $\mu_c(D)$ and η in $\{\eta \in \Xi \mid \eta + \mu_C\ (D) \subset \Xi \cap C^*\}$ such that $\xi_C + \eta = 0$. Hence

$$\mu_C\ (D) - \xi_C \subset \Xi \cap C^*.$$

On the other hand since, for ξ in $\Xi \cap C^*$,

$$\phi_i\ (\xi_c + \zeta) \geqq -m_i$$

for every ϕ_i in $C \cap Sk^1\,(\boldsymbol{C})$, we have

$$\mu_c\ (D) \supset \xi_c + \Xi \cap C^*.$$

Therefore

$$\mu_c\ (D) = \xi_c + \Xi \cap C^*.$$

Hence

$$M_c = A_c e\,(\xi_c).$$

This implies (2).

The equivalence of (2) and (3) is obvious.

REMARK 3.2. *If X is non-singular, then any T-stable Weil divisor is Cartier.*

In fact, for every C in $\boldsymbol{C}$, C is regular. So the equations

$$\phi_i\ (\xi) = -m_i$$

for every ϕ_i in $C \cap Sk^1\,(\boldsymbol{C})$ is always solvable in Ξ.

Now we state the following theorem without proof.

THEOREM 3.3 (Demazure) *There is an exact sequence*

$$0 \to \Xi \overset{j}{\to} \bigoplus_{i=1}^{d} \boldsymbol{Z} D_i \overset{p}{\to} Pic\,(X) \to 0$$

where j sends $\xi \in \Xi$ to $div\,(e(\xi)) = \sum_{i=1}^{d} \phi_i(\xi) D_i$ and p sends D_i to $O_X\,(D_i)$. In particular every line *bundle L on X is isomorphic to* $O_X(\sum_{i=1}^{d} m_i D_i)$ *for some* m_i.

CORORALLY 3.4. *Pic* $(X) \simeq \boldsymbol{Z}^{d-\dim(X)}$

LEMMA 3.5. $\Omega_X^n \simeq O_X(-\sum_{i=1}^{d} D_i)$.

Proof. The rational n-form of $k(T)$ which is T-invariant is unique up to scalar, i. e.

$$\omega = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$$

where $t_i = e(\xi_i)$, ξ_i are base of Ξ.

LEMMA 3.6. *Let* U_c *be an affine open subset of* X *corresponding to a cone* C *in* $\boldsymbol{C}$. *Then*

$$H^0(U_c, O_X(\sum_{i=1}^{d} m_i D_i)) = \bigoplus_{\substack{\phi_i(\xi) \geqq -m_i \\ \phi_i \in C \cap Sk^1(C)}} ke(\xi).$$

In particular

$$H^0(X, O_X(\sum_{i=1}^{d} m_i D_i)) = \bigoplus_{\substack{\phi_i(\xi) \geqq -m_i \\ \phi_i \in Sk^1(C)}} ke(\xi).$$

PROPOSITION 3.7. *Put* $D = \sum_{i=1}^{d} m_i D_i$ *and* $\lambda(D) = \{\xi \in \Xi \mid \phi_i(\xi) \geqq -m_i,\ \phi_i \in Sk^1(C)\}$. *Then*

(1) *OX(D) is generated by its global sections.*

(2) *For every* C *in* $\boldsymbol{C}$, *there exists* ξ_c *in* $\lambda(D)$ *such that*

$$\phi_i(\xi_c) = -m_i \text{ for } \phi_i \text{ in } C \cap Sk^1(C).$$

are equivalent

proof. (1)⇨(2) Since D is a Cartier divisor, for every C in $\boldsymbol{C}$, there exists η_c in Ξ such that

$$\phi_i(\eta_c) = -m_i$$

for every ϕ_i in $C \cap Sk^1(C)$ and

$$\mu_c(D) = \eta_c + \Xi \cap C^*,$$

But by the assumption we have

$$\mu_c(D) = \lambda(D) + \Xi \cap C^*.$$

Since η_c is in $\mu_c(D)$, write

$$\eta_c = \xi_c + \zeta_c$$

for ξ_c in $\lambda(D)$ and ζ_c in $\Xi \cap C^*$. For every ϕ_i in $C \cap Sk^1(C)$

$$-m_i = \phi_i(\eta_c) = \phi_i(\xi_c) + \phi_i(\zeta_c).$$

Since $\phi_i(\xi_c) \geqq -m_i$ and $\phi_i(\zeta_c) \geqq 0$ we have

$$\phi_i(\xi_c) = -m_i.$$

(2)⇨(1) Since

$$\mu_c(D) = \xi_c + \Xi \cap C^*$$

and

$$\mu_c(D) \supset \lambda(D) + \Xi \cap C^* \supset \xi_c + \Xi \cap C^*$$

we have

$$\mu_c(D) = \lambda(D) + \Xi \cap C^*.$$

This means that $O_X(D)$ is generated by $ke(\lambda(D))$.

For the ampleness of line bundle, we state the following theorem without proof.

THEOREM 3.8. (Oda) *Let (X, T) be a complete normal almost homogeneous variety. Put $D = \sum_{i=1}^{d} m_i D_i$. Then the following two conditions are equivalent.*

(1) *$O_X(D)$ is ample.*

(2) *There exists a positive integer n such that, for every maximal cone C in $\boldsymbol{C}$, there exists ξ_c in Ξ so that*

$$\phi_i(\xi_c) = -nm_i \text{ for } \phi_i \in C \cap Sk^1(\boldsymbol{C})$$

$$\phi_j(\xi_c) > -nm_j \text{ for } \phi_j \notin C \cap Sk^1(\boldsymbol{C}).$$

if ξ_c exists then it is unique.

COROLLARY 3.9. (Demazure) *Let (X, T) be a complete non-singular almost homogeneous variety. Put $D = \sum_{i=1}^{d} m_i D_i$. Then the following conditions are equivalent.*

(1) *D is very ample.*

(2) *D is ample.*

(3) *For every maximal cone C in $\boldsymbol{C}$, the unique ξ_c in Ξ such that $\phi_i(\xi_c) = -m_i$ for $\phi_i \in C \cap Sk^1(\boldsymbol{C})$ satisfies $\phi_j(\xi_c) > -m_j$ for $\phi_j \notin C \cap Sk^1(\boldsymbol{C})$.*

§4. Equivariant vector bundles of rank 2 on a ruled surface.

In this section we assume that X is a rational ruled surface F_m. F_m is defined by the cone complex $(\phi_1, \phi_2, \phi_3 = -\phi_1 - m\phi_2, \phi_4 = -\phi_2)$, i.e. $C_i = \boldsymbol{Q}_o\phi_i + \boldsymbol{Q}_o\phi_{i+1}$ $(\phi_5 = \phi_1)$ are two-dimensional cones and $D_i = \boldsymbol{Q}_o\phi_i$ are one-dimensional cones where $\boldsymbol{Q}_o$ is the set of non-negative rational numbers. We denote by the same letters the T-stable divisors of X corresponding to the cone D_i . Let U_i = Spec (A_i) be the T-stable affine open subset of X corresponding to C_i. Let (X_{ij}) be a local equiation of D_i on U_j. Put

$$x = \frac{X_{1j}}{X_{3j}}, \quad y = \frac{X_{2j}}{X^m{}_{3j} X_{4j}}$$

We can easily see that x, y are independent of j without unit. So we take (X_{ij}) so that x, y are independent of j. Then

$$U_1 = \text{Spec}\,(k[x, y])$$

$$U_2 = \text{Spec}\,(k[\tfrac{1}{x}, \tfrac{y}{x^m}])$$

$$U_3 = \text{Spec}\,(k[\tfrac{1}{x}, \tfrac{x^m}{y}])$$

$$U_4 = \text{Spec}\,(k[x, \tfrac{1}{y}])$$

From now on we consider indecomposable equivariant vector bundles of rank 2 on F_m. At first we show examples of them.

(I) Let E_{I} be an equivariant vector bundle defined by the exact sequence

$$0 \to O_{F_m} \xrightarrow{f} O_{F_m}(aD_1 + bD_2) \oplus O_{F_m}(cD_3) \oplus O_{F_m}(dD_4) \to E_{\mathrm{I}}^* \to 0$$

where a, b, c, d are positive integers and f sends 1 to $(X_{1j}^a X_{2j}^b,\ X_{3j}^c,\ X_{4j}^d)$ on U_j. We consider the data (m, P) for E_{I}. Let e_{ij} be defined by

$$O_{F_m}(-aD_1 - bD_2)e_{1j} \oplus O_{F_m}(-cD_3)e_{2j} \oplus O_{F_m}(-dD_4)e_{3j} \xrightarrow{f^*} O_{F_m}$$

on U_j where f^* is the dual of f. We take semi-invariant basis (u_i, v_i) of E_{I} on U_i as follows

$$u_1 = -\frac{e_{11}}{X_{31}^a X_{31}^{bm} X_{41}^b} + \left[\frac{X_{11}}{X_{31}}\right]^a \left[\frac{X_{21}}{X_{41}X_{31}^m}\right]^b \frac{e_{31}}{X_{41}^d}, \quad v_1 = \frac{e_{21}}{X_{31}^c} - \frac{e_{31}}{X_{41}^d},$$

$$u_2 = -\frac{e_{12}}{X_{42}^b X_{12}^{bm} X_{12}^a} + \left[\frac{X_{22}}{X_{42}X_{12}^m}\right]^b \frac{e_{32}}{X_{42}^d}, \quad v_2 = \frac{e_{22}}{X_{12}^c} - \left[\frac{X_{32}}{X_{12}}\right]^c \frac{e_{32}}{X_{42}^d},$$

$$u_3 = \frac{X_{13}^{md}}{X_{23}^d} e_{33} - \left[\frac{X_{43}X_{13}^m}{X_{23}}\right]^d \frac{e_{13}}{X_{13}^a X_{23}^b}, \quad v_3 = \frac{e_{23}}{X_{13}^c} - \left[\frac{X_{33}}{X_{13}}\right]^c \frac{e_{13}}{X_{13}^a X_{23}^b},$$

$$u_4 = \frac{X_{34}^{md}}{X_{24}^d} e_{34} - \left[\frac{X_{44}X_{34}^m}{X_{24}}\right]^d \frac{e_{24}}{X_{34}^c}, \quad v_4 = -\frac{e_{14}}{X_{34}^a X_{24}^b} + \left[\frac{X_{14}}{X_{34}}\right]^a \frac{e_{24}}{X_{34}^c}.$$

Then

$$\begin{bmatrix} x^{-a}y^{-a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{y}{x^m}\right)^{-b} & 0 \\ 0 & x^c \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix},$$

$$\begin{bmatrix} \left(\frac{y}{x^m}\right)^{-b} & 0 \\ 0 & x^c \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{x^m}{y}\right)^{-d} & 0 \\ 0 & x^c \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix},$$

$$\begin{bmatrix} \left(\frac{x^m}{y}\right)^{-d} & 0 \\ 0 & x^c \end{bmatrix} \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^{-a} & 0 \\ 0 & y^d \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \end{bmatrix},$$

$$\begin{bmatrix} x^{-a} & 0 \\ 0 & y^d \end{bmatrix} \begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x^{-a}y^{-a} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \end{bmatrix}.$$

Therefore we can take P of the data (m, P) for E_{I} as follows

$$P(C_1, C_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P(C_2, C_3) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, P(C_3, C_4) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P(C_4, C_1) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$

Since $\phi_1(x) = \phi_2(y) = 1$, $\phi_2(x) = \phi_1(y) = 0$ we can take $x^\alpha y^\beta$ as a character of T. Let ξ_i, η_i be the characters corresponding to u_i, v_i respectively i. e. for example $\xi_1 = x^a y^a . \eta_1 = 1$. Then

$\phi_1(\xi_1) = \phi_1(\eta_4) = a,\ \phi_1(\eta_1) = \phi_1(\xi_4) = 0$

$\phi_2(\xi_1) = \phi_2(\xi_2) = b,\ \phi_2(\eta_1) = \phi_2(\eta_2) = 0$

$\phi_3(\xi_2) = \phi_3(\xi_3) = 0,\ \phi_3(\eta_2) = \phi_3(\eta_3) = c$

$\phi_4(\xi_3) = \phi_4(\xi_4) = d,\ \phi_4(\eta_3) = \phi_4(\eta_4) = 0$

These integers means the integers of m of the data (m, P) for E_{I} i. e.

$m(\phi_1) = (a, 0)$, $m(\phi_2) = (b, 0)$, $m(\phi_3) = (0, c)$, $m(\phi_4) = (d, 0)$.

(II) Let E_{II} be an equivariant vector bundle defined by the exact sequence

$$0 \to O_{F_m} \xrightarrow{f} O_{F_m}(aD_1) \oplus O_{F_m}(bD_2 + cD_3) \oplus O_{F_m}(dD_4) \to E_{\mathrm{II}}^* \to 0$$

where a, b, c, d are positive integers and f sends 1 to $(X_{1j}^a,\ X_{2j}^b X_{3j}^c,\ X_{4j}^d)$ on U_j. Let e_{ij} be the same as in I. We take semi-invariant basis (u_i, v_i) of E_{II} on U_i as follows

$$u_1=\frac{e_{21}}{X_{41}^bX_{31}^{bm}X_{31}^c}-\left[\frac{X_{21}}{X_{41}X_{31}^m}\right]^b\frac{e_{31}}{X_{41}^d},\quad v_1=\frac{e_{11}}{X_{31}^a}-\left[\frac{X_{11}}{X_{31}}\right]^a\frac{e_{31}}{X_{41}^d},$$

$$u_2=\frac{e_{22}}{X_{12}^cX_{42}^bX_{12}^{bm}}-\left[\frac{X_{22}}{X_{42}X_{12}^m}\right]^b\left[\frac{X_{32}}{X_{12}}\right]^c\frac{e_{32}}{X_{42}^b},\quad v_2=\frac{e_{12}}{X_{12}^a}-\frac{e_{32}}{X_{42}^d},$$

$$u_3=\frac{e_{23}}{X_{23}^bX_{13}^c}-\left[\frac{X_{33}}{X_{13}}\right]^c\frac{e_{13}}{X_{13}^a},\quad v_3=-\frac{X_{13}^{md}}{X_{23}^d}e_{33}+\left[\frac{X_{43}X_{13}^m}{X_{23}}\right]^d\frac{e_{13}}{X_{13}^a},$$

$$u_3=-\frac{e_{14}}{X_{34}^a}+\left[\frac{X_{14}}{X_{34}}\right]^a\frac{e_{24}}{X_{24}^bX_{34}^c},\quad v_4=-\frac{X_{34}^{md}}{X_{24}^d}e_{34}+\left[\frac{X_{44}X_{34}^m}{X_{24}}\right]^d\frac{e_{24}}{X_{24}^bX_{34}^c},$$

Then

$$\begin{bmatrix} y^{-b} & 0\\ 0 & x^{-a}\end{bmatrix}\begin{bmatrix}u_1\\ v_1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{y}{x^m}\right)^{-b}x^c & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_2\\ v_2\end{bmatrix},$$

$$\begin{bmatrix}\left(\frac{y}{x^m}\right)^{-b}x^c & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_2\\ v_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}x^c & 0\\ 0 & \left(\frac{x^m}{y}\right)^{-d}\end{bmatrix}\begin{bmatrix}u_3\\ v_3\end{bmatrix},$$

$$\begin{bmatrix}x^c & 0\\ 0 & \left(\frac{x^m}{y}\right)^{-d}\end{bmatrix}\begin{bmatrix}u_3\\ v_3\end{bmatrix}=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}x^{-a} & 0\\ 0 & y^d\end{bmatrix}\begin{bmatrix}u_4\\ v_4\end{bmatrix},$$

$$\begin{bmatrix}x^{-a} & 0\\ 0 & y^d\end{bmatrix}\begin{bmatrix}u_4\\ v_4\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}y^{-b} & 0\\ 0 & x^{-a}\end{bmatrix}\begin{bmatrix}u_1\\ v_1\end{bmatrix}.$$

Hence the data (m, P) for E_{II} are

$$P(C_1, C_2)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, P(C_2, C_3)=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, P(C_3, C_4)=\begin{bmatrix}1 & 0\\ -1 & 1\end{bmatrix}, P(C_4, C_1)=\begin{bmatrix}1 & -1\\ 1 & 0\end{bmatrix}$$

and

$$m(\phi_1)=(0, a),\ m(\phi_2)=(b, 0),\ m(\phi_3)=(c, 0),\ m(\phi_4)=(0, d)$$

by the same way as I.

(III) Let E_{III} be an equivariant vector bundle defined by the evact sequence

$$0\to O_{F_m}\xrightarrow{f}O_{F_m}(aD_1)\oplus O_{F_m}(bD_2)\oplus O_{F_m}(cD_3+dD_4)\to E_{III}^*\to 0$$

where a, b, c, d are positive integers and f sends 1 to $(X_{1j}^a,\ X_{2j}^b,\ X_{3j}^c X_{4j}^d)$ on U_j. Let e_{ij} be the same as in I. We take semi-invariant basis (u_i, v_i) of E_{III} on U_i as follows

$$u_1=-\frac{e_{21}}{X_{41}^bX_{31}^{bm}}+\left[\frac{X_{21}}{X_{41}X_{31}^m}\right]^b\frac{e_{31}}{X_{31}^cX_{41}^d},\quad v_1=-\frac{e_{11}}{X_{31}^a}+\left[\frac{X_{11}}{X_{31}}\right]^a\frac{e_{31}}{X_{31}^cX_{41}^d},$$

$$u_2=-\frac{e_{22}}{X_{42}^bX_{12}^{bm}}+\left[\frac{X_{22}}{X_{42}X_{12}^m}\right]^b\frac{e_{12}}{X_{12}^a},\quad v_2=\frac{e_{32}}{X_{12}^cX_{42}^d}-\left[\frac{X_{32}}{X_{12}}\right]^c\frac{e_{12}}{X_{12}^a},$$

$$u_3=-\frac{e_{23}}{X_{23}^b}+\frac{e_{13}}{X_{13}^a},\quad v_3=\frac{X_{13}^{md}}{X_{23}^dX_{13}^c}e_{33}-\left[\frac{X_{33}}{X_{13}}\right]^c\left[\frac{X_{43}X_{13}^m}{X_{23}}\right]^d\frac{e_{13}}{X_{13}^a},$$

$$u_4=\frac{e_{14}}{X_{34}^a}-\left[\frac{X_{14}}{X_{34}}\right]^a\frac{e_{24}}{X_{24}^b},\quad v_4=\frac{X_{34}^{md}}{X_{24}^dX_{34}^c}e_{34}-\left[\frac{X_{44}X_{34}^m}{X_{24}}\right]^d\frac{e_{24}}{X_{24}^b}.$$

Then

$$\begin{bmatrix} y^{-b} & 0\\ 0 & x^{-a}\end{bmatrix}\begin{bmatrix}u_1\\ v_1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}\left(\frac{y}{x^m}\right)^{-b} & 0\\ 0 & x^c\end{bmatrix}\begin{bmatrix}u_2\\ v_2\end{bmatrix},$$

$$\begin{bmatrix} \left(\frac{y}{x^m}\right)^{-b} & 0 \\ 0 & x^c \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & x^c\left(\frac{x^m}{y}\right)^{-d} \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 \\ 0 & x^c\left(\frac{x^m}{y}\right)^{-d} \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x^{-a} & 0 \\ 0 & y^d \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \end{bmatrix},$$

$$\begin{bmatrix} x^{-a} & 0 \\ 0 & y^d \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} y^{-b} & 0 \\ 0 & x^{-a} \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}.$$

Hence the data (m, P) for E_{III} are

$$P(C_1, C_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P(C_2, C_3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P(C_3, C_4) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, P(C_4, C_1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$m(\phi_1) = (0, a), m(\phi_2) = (b, 0), m(\phi_3) = (0, c), m(\phi_4) = (0, d)$$

by the same way as I.

(IV) Let E_{IV} be an equivariant vector bundle defined by the exact sequence

$$0 \to O_{F_m} \xrightarrow{f} O_{F_m}(dD_4 + aD_1) \oplus O_{F_m}(bD_2) \oplus O_{F_m}(cD_3) \to E^*_{\mathrm{IV}} \to 0$$

where a, b, c, d are positive integers and f sends 1 to $(X_{4j}^d X_{1j}^a, \ X_{2j}^b, \ X_{3j}^c)$ on U_j. Let e_{ij} be the same as in I. We take semi-invariant basis (u_i, v_i) of E_{IV} on U_i as follows

$$u_1 = \frac{e_{21}}{X_{41}^b X_{32}^{mb}} - \left[\frac{X_{21}}{X_{41}X_{31}^m}\right]^b \frac{e_{31}}{X_{31}^c}, \quad v_1 = \frac{e_{11}}{X_{31}^a X_{41}^d} - \left[\frac{X_{11}}{X_{31}}\right]^a \frac{e_{31}}{X_{31}^c},$$

$$u_2 = \frac{e_{22}}{X_{41}^b X_{12}^{mb}} - \left[\frac{X_{22}}{X_{42}X_{12}^m}\right]^b \frac{e_{12}}{X_{12}^a X_{42}^d}, \quad v_2 = -\frac{e_{32}}{X_{12}^c} + \left[\frac{X_{32}}{X_{12}}\right]^c \frac{e_{12}}{X_{12}^a X_{42}^d},$$

$$u_3 = -\frac{X_{13}^{md}}{X_{23}^d X_{13}^a} e_{13} + \left[\frac{X_{43}X_{13}^m}{X_{23}}\right]^d \frac{e_{23}}{X_{23}^b}, \quad v_3 = -\frac{e_{33}}{X_{13}^c} + \left[\frac{X_{33}}{X_{13}}\right]^c \frac{e_{23}}{X_{23}^b},$$

$$u_4 = -\frac{X_{34}^{md}}{X_{34}^a X_{24}^d} e_{14} + \left[\frac{X_{14}}{X_{34}}\right]^a \left[\frac{X_{44}X_{34}^m}{X_{24}}\right]^d \frac{e_{24}}{X_{24}^b}, \quad v_4 = -\frac{e_{34}}{X_{34}^c} + \frac{e_{24}}{X_{24}^b}.$$

Then

$$\begin{bmatrix} x^{-a} & 0 \\ 0 & y^{-b} \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \left(\frac{y}{x^m}\right)^{-b} & 0 \\ 0 & x^c \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{y}{x^m}\right)^{-b} & 0 \\ 0 & x^c \end{bmatrix}\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \left(\frac{x^m}{y}\right)^{-d} & 0 \\ 0 & x^c \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{x^m}{y}\right)^{-d} & 0 \\ 0 & x^c \end{bmatrix}\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x^{-a}y^d & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \end{bmatrix}$$

$$\begin{bmatrix} x^{-a}y^d & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x^{-a} & 0 \\ 0 & y^{-b} \end{bmatrix}\begin{bmatrix} u_1 \\ v_1 \end{bmatrix}$$

Hence the data (m, P) for E_{IV} are

$$P(C_1, C_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P(C_2, C_3) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, P(C_3, C_4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P(C_4, C_1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

and

$$m(\phi_1) = (0, a), m(\phi_2) = (b, 0), m(\phi_3) = (0, c), m(\phi_4) = (d, 0)$$

by the same way as I.

Next we consider the data (m, P) for an equivariant indecomposable vector bundle of rank 2 on F_m. At first we deal the case when m of the data (m, P) satisfies

$$m_{c_1}(\phi_2)=m_{c_2}(\phi_2)=(\alpha_2, 0)$$

and one of

(1) $m_{c_2}(\phi_3)=m_{c_3}(\phi_3)=(\alpha_3, 0)$

(2) $m_{c_2}(\phi_3)=m_{c_3}(\phi_3)=(0, \alpha_3)$

and one of

(α) $m_{c_3}(\phi_4)=m_{c_4}(\phi_4)=(\alpha_4, 0)$

(β) $m_{c_3}(\phi_4)=m_{c_4}(\phi_4)=(0, \alpha_4)$

and one of

(a) $m_{c_4}(\phi_1)=m_{c_1}(\phi_1)=(\alpha_1, 0)$

(b) $m_{c_4}(\phi_1)=m_{c_1}(\phi_1)=(0, \alpha_1)$

(c) $m_{c_4}(\phi_1)=(\alpha_1, 0),\ m_{c_1}(\phi_1)=(0, \alpha_1)$

(d) $m_{c_4}(\phi_1)=(0, \alpha_1),\ m_{c_1}(\phi_1)=(\alpha_1, 0)$

where α_i are non-negative integers. If one of α_i is zero then we may assume $\alpha_1=0$. In this case P of the data (m, P) are

$$P(C_1, C_2)=\begin{bmatrix} a & b \\ 0 & d \end{bmatrix},$$

(1) $P(C_2, C_3)=\begin{bmatrix} a' & b' \\ 0 & d' \end{bmatrix}$, (2) $P(C_2, C_3)=\begin{bmatrix} a' & 0 \\ c' & d' \end{bmatrix}$,

(α) $P(C_3, C_4)=\begin{bmatrix} a'' & b'' \\ 0 & d'' \end{bmatrix}$, ($\beta$) $P(C_3, C_4)=\begin{bmatrix} a'' & 0 \\ c'' & d'' \end{bmatrix}$,

$$P(C_4, C_1)=\begin{bmatrix} a''' & b''' \\ c''' & d''' \end{bmatrix}.$$

Case 1, α). In the equivalence relation we take

$$\rho(C_1)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \rho(C_2)=\begin{bmatrix} 1 & -\frac{b}{a} \\ 0 & 1 \end{bmatrix},\quad \rho(C_3)=\begin{bmatrix} 1 & -\frac{ab'+bd'}{aa'} \\ 0 & 1 \end{bmatrix},$$

$$\rho(C_4)=\begin{bmatrix} 1 & -\frac{aa'b''+(ab'+bd')d''}{aa'a''} \\ 0 & 1 \end{bmatrix}$$

Then

$$P'(C_1, C_2)=\rho(C_1)^{-1}P(C_1, C_2)\,\rho(C_2)=\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$

$$P'(C_2, C_3)=\rho(C_2)^{-1}P(C_2, C_3)\,\rho(C_3)=\begin{bmatrix} a' & 0 \\ 0 & d' \end{bmatrix}$$

$$P'(C_3, C_4)=\rho(C_3)^{-1}P(C_3, C_4)\,\rho(C_4)=\begin{bmatrix} a'' & 0 \\ 0 & d'' \end{bmatrix}$$

From the relation

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we have

$$P'(C_4,\ C_1)=\begin{bmatrix} a^{-1}a'^{-1}a''^{-1} & 0 \\ 0 & d^{-1}d'^{-1}d''^{-1}\end{bmatrix}$$

Therefore this case defines an decomposable vector bundle.

Case 1, β). We take

$$\rho(C_1)=\begin{bmatrix} 1 & \dfrac{ab'+bd'}{dd'} \\ 0 & 1\end{bmatrix},\ \rho(C_2)=\begin{bmatrix} 1 & \dfrac{b'}{d'} \\ 0 & 1\end{bmatrix},\ \rho(C_3)=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},$$
$$\rho(C_4)=\begin{bmatrix} 1 & 0 \\ -\dfrac{c''}{d''} & 1\end{bmatrix}.$$

Then the vector bundle $E(m,\ P)$ defined by the data $(m,\ P)$ is decomposable.

Case 2, α). We take

$$\rho(C_1)=\begin{bmatrix} 1 & \dfrac{b}{d} \\ \dfrac{dc'}{aa'} & 1+\dfrac{bc'}{aa'}\end{bmatrix},\ \rho(C_2)=\begin{bmatrix} 1 & 0 \\ \dfrac{c'}{d'} & 1\end{bmatrix},\ \rho(C_3)=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},\ \rho(C_4)=\begin{bmatrix} 1 & -\dfrac{b''}{a''} \\ 0 & 1\end{bmatrix}.$$

Then $E(m,\ P)$ is decomposable.

Case 2, β). We take

$$\rho(C_1)=\begin{bmatrix} 1 & \dfrac{b}{d} \\ 0 & 1\end{bmatrix},\ \rho(C_2)=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},\ \rho(C_3)=\begin{bmatrix} 1 & 0 \\ -\dfrac{c'}{d'} & 1\end{bmatrix},\ \rho(C_4)=\begin{bmatrix} 1 & 0 \\ -\dfrac{c'a''+c''d''}{d'd''} & 1\end{bmatrix}.$$

Then $E(m,\ P)$ is decomposable.

So we may assume that α_i is a positive integer.

Furthermore by taking

$$\rho(C_1)=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix},\ \rho(C_2)=\begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1}\end{bmatrix},\ \rho(C_3)=\begin{bmatrix} a^{-1}a'^{-1} & 0 \\ 0 & d^{-1}d'^{-1}\end{bmatrix},$$
$$\rho(C_4)=\begin{bmatrix} a^{-1}a'^{-1}a''^{-1} & 0 \\ 0 & d^{-1}d'^{-1}d''^{-1}\end{bmatrix}$$

we may assume that

$$P(C_1,C_2)=\begin{bmatrix} 1 & b \\ 0 & 1\end{bmatrix},$$

(1) $P(C_2, C_3)=\begin{bmatrix} 1 & b' \\ 0 & 0\end{bmatrix}$, (2) $P(C_2, C_3)=\begin{bmatrix} 1 & 0 \\ c' & 1\end{bmatrix}$,

(α) $P(C_3, C_4)=\begin{bmatrix} 1 & b'' \\ 0 & 1\end{bmatrix}$, ($\beta$) $P(C_3, C_4)=\begin{bmatrix} 1 & 0 \\ c'' & 1\end{bmatrix}$,

and

(a) $P(C_4, C_1)=\begin{bmatrix} a''' & b''' \\ 0 & d'''\end{bmatrix}$, $a'''d'''=1$,

(b) $P(C_4, C_1)=\begin{bmatrix} a''' & 0 \\ c''' & d'''\end{bmatrix}$, $a'''d'''=1$,

(c) $P(C_4, C_1)=\begin{bmatrix} a''' & d''' \\ c''' & 0\end{bmatrix}$, $b'''d'''=-1$,

(d) $P(C_4, C_1)=\begin{bmatrix} 0 & b''' \\ c''' & d'''\end{bmatrix}$, $b'''d'''=-1$.

Under these assumption,

LEMMA 4.1. *In the case (1, α, c), (1, α, d), (1, β, d) and (2, β, d) there is no equivariant vector* *bundle.*

proof. Case 1, α, c). In this case

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1) = \begin{bmatrix} a''' + c'''(b+b'+b'') & b''' \\ c''' & 0 \end{bmatrix}.$$

But this contradicts to the relation

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So this case cannot happen.

Case 1, α, d). In this case

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1) = \begin{bmatrix} c'''(b+b'+b'') & b''' + d'''(b+b'+b'') \\ c''' & d''' \end{bmatrix}.$$

c'''must be zero. Then this is a contradiction.

Case 1, β, d).

$$P(C_1, C_2)\,P(C_2,C_3)\,P(C_3, C_4)\,P(C_4, C_1) = \begin{bmatrix} c'''(b+b') & b'''\{1 + c''(b+b')\} + d'''(b+b') \\ c''' & c''b''' + d''' \end{bmatrix}.$$

Case 2, β, d).

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1) = \begin{bmatrix} bc''' & b'''\{1 + b(c'+c'')\} + bd''' \\ c''' & b'''(c'+c'') + d''' \end{bmatrix}.$$

So these cases cannot happen.

LEMMA 4.2. *The case $(1, \alpha, a)$, $(1, \alpha, b)$, $(1, \beta, a)$, $(1, \beta, b)$, $(1, \alpha, a)$, $(2, \alpha, b)$, $(2, \beta, a)$ and $(2, \beta, b)$ determine decomposable vector bundles.*

proof. By the equivalence relation we may show that $P(C_i, C_{i+1})$ are equivalent to $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Case 1, α, a) and Case 1, α, b). We take

$$\rho(C_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \rho(C_2) = \begin{bmatrix} 1 & -b \\ 0 & 1 \end{bmatrix},\ \rho(C_3) = \begin{bmatrix} 1 & -b-b' \\ 0 & 1 \end{bmatrix},\ \rho(C_4) = \begin{bmatrix} 1 & -b-b'-b'' \\ 0 & 1 \end{bmatrix}.$$

Then

$$P'(C_i, C_{i+1}) = \rho(C_i)^{-1}\,\rho(C_i, C_{i+1})\,\rho(C_i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad i=1, 2, 3.$$

From the relation

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we have

$$P'(C_4, C_1) = \rho(C_4)^{-1}\,P(C_4, C_1)\,\rho(C_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Case 1, β, a) Since

$$P(C_1, C_2)\,P(C_2, C_3)\,P(C_3, C_4)\,P(C_4, C_1)$$

$$= \begin{bmatrix} a'''\{1 + c''(b+b')\} + d'''(b+b') & b'''\{1 + c''(b+b')\} + d'''(b+b') \\ c''a''' & c''b''' + d''' \end{bmatrix}$$

we have

$$c''=0,\ d'''=1,\ a'''=1,\ b+b'=0,\ b'''=0.$$

We take

$$\rho(C_1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_2)=\begin{pmatrix}1&b'\\0&1\end{pmatrix},\ \rho(C_3)=\rho(C_4)=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Then

$$P'(C_i,\ C_{i+1})=\rho(C_i)^{-1}P(C_i,\ C_{i+1})\,\rho(C_{i+1})=\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

for $i=1, 2, 3, 4$ where $C_5=C_1$.

Case 1, β, b) Since

$$P(C_1,\ C_2)P(C_2,C_3)P(C_3,\ C_4)P(C_4,\ C_1)=\begin{pmatrix}a'''+(b+b')(c''a'''+c''') & (b+b')d'''\\ c''a'''+c''' & d'''\end{pmatrix}$$

we have

$$c''a'''+c'''=0,\ d'''=1,\ a'''=1,\ b+b'=0.$$

we take

$$\rho(C_1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_2)=\begin{pmatrix}1&b'\\0&1\end{pmatrix},\ \rho(C_3)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_4)=\begin{pmatrix}1&0\\-c''&1\end{pmatrix}.$$

Then

$$P'(C_i,\ C_{i+1})=\rho(C_i)^{-1}P(C_i,\ C_{i+1})\,\rho(C_{i+1})=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ i=1,\ 2,\ 3,\ 4.$$

Other cases are dealt by the same way. So we only show $\rho(C_i)$.

Case 2, α, a)

$$\rho(C_1)=\begin{pmatrix}1&b\\0&1\end{pmatrix},\ \rho(C_2)=\rho(C_3)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_4)=\begin{pmatrix}1&b+b'''\\0&1\end{pmatrix}.$$

Case 2, α, b)

$$\rho(C_4)=\rho(C_1)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_2)=\begin{pmatrix}1&-b\\0&1\end{pmatrix},\ \rho(C_3)=\begin{pmatrix}1+bc'&b\\c'&1\end{pmatrix}^{-1}.$$

Case 2, β, a)

$$\rho(C_1)=\begin{pmatrix}1&-b\\0&1\end{pmatrix},\ \rho(C_2)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_3)=\begin{pmatrix}1&0\\-c'&1\end{pmatrix},\ \rho(C_4)=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

Case 2, β, b)

$$\rho(C_1)=\rho(C_2)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \rho(C_3)=\begin{pmatrix}1&0\\-c'&1\end{pmatrix},\ \rho(C_4)=\begin{pmatrix}1&0\\c''&1\end{pmatrix}.$$

Therefore these cases determine a decomposable vector bundles.

LEMMA 4.3. *The case (1, β, c), (2, α, c), (2, α, d) and (2, β, c) determine an indecomposable vector bundles of rank 2. Furthermore P of (m, P) are equivalent to*

$$(1,\ \beta,\ \mathrm{c});\ P(C_1,\ C_2)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ P(C_2,\ C_3)=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ P(C_3,\ C_4)=\begin{pmatrix}1&0\\-1&1\end{pmatrix},$$

$$P(C_4,\ C_1)=\begin{pmatrix}1&-1\\1&0\end{pmatrix},$$

$$(2,\ \alpha,\ \mathrm{c});\ P(C_1,\ C_2)=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ P(C_2,\ C_3)=\begin{pmatrix}1&0\\-1&1\end{pmatrix},\ P(C_3,\ C_4)=\begin{pmatrix}1&0\\0&1\end{pmatrix},$$

$$P(C_4,\ C_1)=\begin{pmatrix}1&-1\\1&0\end{pmatrix},$$

$$(2, \alpha, d); P(C_1, C_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P(C_2, C_3) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, P(C_3, C_4) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

$$P(C_4, C_1) = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix},$$

$$(2, \beta, c); P(C_1, C_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, P(C_2, C_3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P(C_3, C_4) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$P(C_4, C_1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

proof. Case 1, β, c). Since

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_4)P(C_4, C_1) = \begin{bmatrix} a'''\{1+c''(b+b')\}+c'''(b+b') & b'''\{1+c''(b+b')\} \\ c''a'''+c''' & c''b''' \end{bmatrix}$$

we have

$$b'''\ \{1+c''(b+b')\}=c''a'''+c'''=0,$$

$$a'''\ \{1+c''(b+b')\}+c'''(b+b')=c''b'''=1,$$

Hence

$$a'''=1,\ b+b'=-b''',\ c'''=-b'''^{-1},\ c''=b'''^{-1}.$$

We take

$$\rho(C_1) = \begin{bmatrix} -b''' & 0 \\ 0 & 1 \end{bmatrix},\ \rho(C_2) = \begin{bmatrix} -b''' & b'+b''' \\ 0 & 1 \end{bmatrix},\ \rho(C_3) = \rho(C_4) = \begin{bmatrix} -b''' & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\rho(C_1)^{-1}P(C_1, C_2)\rho(C_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\rho(C_2)^{-1}P(C_2, C_3)\rho(C_3) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\rho(C_3)^{-1}P(C_3, C_4)\rho(C_4) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$
$$\rho(C_4)^{-1}P(C_4, C_1)\rho(C_1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Case 2, α, c) Since

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_4)P(C_4, C_1) = \begin{bmatrix} a'''(1+bc')+c'''\{b''(1+bc')+b\} & b'''(1+bc') \\ c'a'''+c'''(1+c'b'') & c'b''' \end{bmatrix}$$

we have

$$c'=b'''^{-1},\ b=-b''',\ c'''=-b'''^{-1},\ a'''=1+b''b'''^{-1}.$$

We take

$$\rho(C_1) = \rho(C_2) = \rho(C_3) = \begin{bmatrix} -b''' & 0 \\ 0 & 1 \end{bmatrix},\ \rho(C_4) = \begin{bmatrix} -b''' & -b'' \\ 0 & 1 \end{bmatrix}.$$

Then

$$\rho(C_1)^{-1}P(C_1, C_2)\rho(C_2) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\rho(C_2)^{-1}P(C_2, C_3)\rho(C_3) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$
$$\rho(C_3)^{-1}P(C_3, C_4)\rho(C_4) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\rho(C_4)^{-1}P(C_4, C_1)\rho(C_1) = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

Case 2, β, d) Since

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_4)P(C_4,C_1)$$

$$=\begin{bmatrix} c'''\{b''(1+bc')+b\} & b'''(1+bc')+d'''\{b''(1+bc')+b\} \\ c'''(1+c'b'') & c'b'''+d'''(1+c'b'') \end{bmatrix}$$

we have

$$c'=b'''^{-1},\ b''=-b''',\ c'''=-b'''^{-1},\ d'''=1+bb'''^{-1}.$$

we take

$$\rho(C_1)=\begin{bmatrix} -b''' & b \\ 0 & 1 \end{bmatrix},\ \rho(C_2)=\rho(C_3)=\rho(C_4)=\begin{bmatrix} -b''' & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\rho(C_1)^{-1}P(C_1, C_2)\rho(C_2)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\rho(C_2)^{-1}P(C_2, C_3)\rho(C_3)=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$
$$\rho(C_3)^{-1}P(C_3, C_4)\rho(C_4)=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\rho(C_4)^{-1}P(C_4, C_1)\rho(C_1)=\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Case 2, β, c). Since

$$P(C_1, C_2)P(C_2, C_3)P(C_3, C_4)P(C_4, C_1)$$

$$=\begin{bmatrix} a'''\{1+b(c'+c'')+bc'''\} & b'''\{1+b(c'+c'')\} \\ a'''(c'+c'')+c''' & b'''(c'+c'') \end{bmatrix}$$

we have

$$b=-1,\ b'''=-b=1,\ c'''=-b'''^{-1}=-1,\ a'''=1,\ c'+c''=-c'''=1.$$

We have

$$\rho(C_4)=\rho(C_1)=\rho(C_2)=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},\ \rho(C_3)=\begin{bmatrix} -1 & 0 \\ c' & 1 \end{bmatrix}.$$

Then

$$\rho(C_1)^{-1}P(C_1, C_2)\rho(C_2)=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\rho(C_2)^{-1}P(C_2, C_3)\rho(C_3)=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho(C_3)^{-1}P(C_3, C_4)\rho(C_4)=\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\rho(C_4)^{-1}P(C_4, C_1)\rho(C_1)=\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

THEOREM 4.4. *Let F_m be a rational ruled surface. F_m becomes an almost homogeneous variety with standard action of T^2. An indecomposable equivariant vector bundle of rank 2 on F_m is isomorphic to $E_J(D)$ for some $D \in Pic(F_m)$ where J is one of* I, II, III, IV *as in the examples.*

proof. Since an equivariant vector bundle has a T-linearization we only show that the data (m, P) defines $E_J(D)$, By tensoring $O_{F_m}(\sum_{i=1}^{4} m_iD_i)$ for some m_i with the vector bundle $E(m, P)$ defined by the data (m, P), we may assume that m of the data (m, P) are the form shown before lemma 4.1. By lemma 4.3 the cases that (m, P) defines an indecomposable equivariant vector bundle are $(1, \beta, c)$, $(2, \alpha, c)$, $(2, \alpha, \mathrm{d})$, and $(2, \beta, \mathrm{c})$. Clearly these cases determine the vector bundles E_{I}, E_{II}, E_{III}, and E_{IV} respectively.

REMARK 4.5. *The same technique will determine an equivariant vector bundle where X is a two-dimensional almost homogeneous variety.*

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