A note on the Jordan decomposition of derivations

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Abstract

As a linear map, a derivation of a K-algebra can be decomposed into semi-simple part and nilpotent part and the each part is also a K-derivation of the K-algebra. If the semi-simple part is not trivial the K-algebra is decomposed into eigenspaces.

§1 Jordan decompositions.

First, we recall the Jordan decomposition of an linear endomorphism of a finite dimensional vector space. Let K be an algebraically closed field and V be a finite dimensional vector space over K. If \( f \) is a K-linear endomorphism of V then there is a semi-simple K-linear endomorphism s and a nilpotent K-linear endomorphism n of V such that

\[ f = s + n \text{ and } sn = ns. \]

Furthermore, n and s are uniquely determined by f. We denote s by \( f_s \) and n by \( f_n \), and we call \( f_s \) the semi-simple part of f and \( f_n \) the nilpotent part of f. Let \( \lambda_1, \ldots, \lambda_n \) be the eigen values of \( f_s \) and let \( V_i = \{ x \in V | f_s(x) = \lambda_i x \} \) for \( i = 1, 2, \ldots, n \). Then \( V_i \) consists of elements x such that \( (f - \lambda_i)^m(x) = 0 \) for some \( m > 0 \). V is a direct sum of \( V_1, \ldots, V_n \).

From now on, we assume that \( \text{ch}(K) = 0 \). We apply the Jordan decomposition to a finite dimensional K-algebra.

Theorem 1. Let A be a finite dimensional K-algebra and D a K-derivation of A. Then \( D_s \) and \( D_n \) are K-derivations of A.

Proof. Put \( S = D_s \) and \( N = D_n \). If \( S = 0 \) there is nothing to prove. Let \( S \neq 0 \). Then A is a direct sum of eigenspaces \( A_\lambda \) of S. Let \( x \in A_\lambda \) and \( y \in A_\mu \). Then \( S(x) = \lambda x \) and \( S(y) = \mu y \). So, there are integers \( n, m > 0 \) such that \( (D - \lambda)^n(x) = 0 \) and \( (D - \mu)^m(y) = 0 \). By the formula

\[ (D - \lambda - \mu)^k(xy) = \sum_{i=0}^{k} \binom{k}{i} (D - \lambda)^i(x)(D - \mu)^{k-i}(y) \]

we get \( (D - \lambda - \mu)^k(xy) = 0 \) for \( k > n + m \). This shows that \( xy \in A_{\lambda + \mu} \). Then,

\[ S(xy) = (\lambda + \mu)xy = (\lambda x)y + x(\mu y) = S(x)y + xS(y). \]

Hence \( S \) is a K-derivation of A.
Let \( (A, m, K) \) be a complete local ring and let \( \text{Der}(A) \) be the set of \( K \)-derivations of \( A \). It is well-known that \( \text{Der}(A) \) is an \( A \)-module and has a structure of a Lie algebra over \( K \). We refer the reader to H. Matsumura's book for the fundamental properties of complete local rings and derivations.

In the sequel \( R \) denotes the formal power series ring \( K[[X_1, \ldots, X_n]] \) in \( X_1, \ldots, X_n \) over \( K \) and \( A=R/I \) for some ideal \( I \) of \( R \). Let \( \partial_i \) be the \( K \)-derivation of \( R \) given by \( \partial_i (X_j) = \delta_{ij} \).

For any \( K \)-derivation \( \partial \) of \( A \) there exist \( A_1, \ldots, A_n \in R \) such that

\[
\partial \left( X \equiv \text{mod} \ I \right) = A_1 \partial_1 (X) + \cdots + A_n \partial_n (X) \equiv \text{mod} \ I
\]

for \( X \equiv R \) and \( \partial \) is denoted by \( a_1 \partial_1 + \cdots + a_n \partial_n \), where \( a_i = A_i \equiv \text{mod} \ I \).

Suppose that \( a_i \in m \) for \( i = 1, \ldots, n \). Then \( \partial \left( m^k \right) \subseteq m^k \) by Leibniz rule. Hence, \( \partial \) induces a \( K \)-derivation \( \partial_k \) of \( A/m^k \) such that

\[
\pi_k \partial = \partial_k \pi_k,
\]

where \( \pi_k : A \rightarrow A/m^k \) is the natural projection.

Applying Theorem 1 to \( \partial_k \), we have the Jordan decomposition

\[
\partial_k = \partial_{k_1} + \partial_{k_2}.
\]

There is a polynomial \( f(X) \in K[X] \) such that \( \partial_{k_1} = f(\partial_k) \) and \( f(0) = 0 \). Since \( \partial_k \) maps the maximal ideal of \( A/m^k \) into itself the same is true for \( \partial_{k_1} \) and \( \partial_{k_2} \). Let \( \phi_k : A/m^k+1 \rightarrow A/m^k \) be the natural projection. Then we have \( \phi_k \partial_{k_1} (m^k/m^k+1) = 0 \) and we have a \( K \)-derivation \( S \) of \( A/m^k \) such that

\[
\phi_k \partial_{k_1} = S \phi_k.
\]

Similarly, we have a \( K \)-derivation \( \mathcal{N} \) of \( A/m^k \) such that

\[
\phi_k \partial_{k_2} = \mathcal{N} \phi_k.
\]

Let \( \lambda_1, \ldots, \lambda_r \) be eigenvalues of \( \partial_{k_1} \) and decompose

\[
A/m^k+1 = A_1 \oplus \cdots \oplus A_r,
\]

where \( A_i = \{ x \in A/m^k+1 | \partial_{k_1} (x) = \lambda_i x \} \).

Then

\[
A/m^k = \sum_{i=1}^r \phi_k (A_i)
\]

For \( a \in A_i \) we have

\[
S(\phi_k (a)) = \phi_k (\partial_{k_1} (a)) = \lambda_i \phi_k (a).
\]

Hence \( S \) is semi-simple and \( \phi_k (A_i) \) is an eigenspace of \( S \).

If \( (\partial_{k_1})^m = 0 \) we have
\[ 0 = \phi_k(\partial_{k+1})^m = N^m \phi_k. \]

Since \( \phi_k \) is onto we have \( N^m = 0 \).

From \( \partial_{k+1} \partial_{k+1} = \partial_{k+1} \partial_{k+1} \) we have
\[ N \phi_k \partial_{k+1} = \phi_k \partial_{k+1} \partial_{k+1} \partial_{k+1} = \phi_k \partial_{k+1} \partial_{k+1} \partial_{k+1} = SN \phi_k. \]

Since \( \phi_k \) is surjective we have \( SN = NS \). Therefore, by the uniqueness of the Jordan decomposition, \( \partial_{m} = S \) and \( \partial_{n} = N \). Now we have
\[ \phi_k \partial_{k+1} = \partial_{k} \phi_k \text{ and } \phi_k \partial_{n+1} = \partial_{k} \phi_k. \]

For \( a \in A \), we put \( a_k = \partial_k \text{ (a mod m^k)} \) and \( b_k = \partial_n \text{ (a mod m^k)} \). Then we have
\[ \phi_k(a_{k+1}) = a_k \text{ and } \phi_k(b_{k+1}) = b_k. \]

Since \( A \) is complete
\[ (0, a_0, a_1, \ldots, a_k, \ldots) \in \varprojlim_k A/m^k = A \text{ and } \]
\[ (0, b_0, b_1, \ldots, b_k, \ldots) \in \varprojlim_k A/m^k. \]

we can define two maps \( \partial_s \) and \( \partial_n \), from \( A \) into itself, by
\[ \partial_s(a) = (0, a_0, a_1, \ldots, a_k, \ldots) \text{ and } \]
\[ \partial_n(a) = (0, b_0, b_1, \ldots, b_k, \ldots). \]

Clearly, \( \partial_s \) and \( \partial_n \) are \( K \)-derivations of \( A \) and \( \partial = \partial_s + \partial_n \). Thus, we have proved:

**Theorem 2.** If \( \partial \) is a \( K \)-derivation of a complete local ring \((A, m, K)\) such that \( \partial(m) \subseteq m \), then there are \( K \)-derivations \( \partial_s \) and \( \partial_n \) of \( A \) such that \( \partial = \partial_s + \partial_n \) and, for \( k = 1, 2, \ldots \), the \( K \)-derivations \( \partial_{s_k} \) and \( \partial_{n_k} \) of \( A/m^k \) induced from \( \partial_s \) and \( \partial_n \) are semi-simple and nilpotent, respectively, and \( \partial_s \partial_n = \partial_n \partial_s \).

§2 Decomposition of complete local rings.

Let \( \partial \) be a \( K \)-derivation of a complete local ring \((A, m, K)\) such that \( \partial(m) \subseteq m \) and \( \partial_{2^k} \neq 0 \). If \( \lambda_0 = 0, \lambda_1, \ldots, \lambda_r \) are eigenvalues of \( \partial_{2^k} \), we can decompose
\[ A/m^2 = A_0 \oplus A_1 \oplus \cdots \oplus A_r, \]
where \( A_i = \{ x \in m/m^2 | \partial_{2^k}(x) = \lambda_i x \} \).

Since \( \partial_{2^k}(K) = 0 \) and \( \partial_{2^k}(m/m^2) \subseteq m/m^2 \), we have \( A_i \subseteq m/m^2 \) if \( \lambda_i \neq 0 \). Fix \( \lambda_i (\lambda_i \neq 0) \). Let
\[ A_{ki} = \{ x \in A/m^k | \partial_{2^k}(x) = \lambda_i x \}. \]

As we have seen in §1, \( \phi_k \) maps \( A_{k+1} \) onto \( A_{k+1} \). For any \( x \in A_1 \), we can find \( a_k \in A_{k+1} \) (\( k = 2, 3, \ldots \)) such that \( a_k = x \), \( a_k = \phi_k(a_{k+1}) \) for \( K = 2, 3, \ldots \). Then
\[ a = (0, a_2, a_3, \ldots) \in \varprojlim_k A/m^k = A. \]
and $\partial_s(a) = \lambda a$.

Applying this argument to a basis of $m/m^2$ we can find minimal generators $x_1, x_2, \ldots, x_n$ of $m$ such that $\partial_s(x_i) = \lambda^{(s)} x_i$, for some eigenvalue $\lambda^{(s)}$ of $\partial_s$. Let $\Lambda \subseteq K$ be an additive semi-group generated by $\lambda_0, \ldots, \lambda_r$. If $x, y \in \Lambda/m^k$ satisfy $\partial_s(x) = \lambda x$, $\partial_s(y) = \mu y$ and $xy > 0$, we have $\partial_s(xy) = (\lambda + \mu)xy$ by the proof of Theorem 1. Let $A_k = \lim_{k \to \infty} A_k$, where $\lambda \in A$ and $A_k = \{x \in \Lambda/m^k | \partial_s(x) = \lambda x\}$. Then

$$A_1 = \{a \in A | \partial_s(a) = \lambda a\} \text{ and } A_1 A_r \subseteq A_{1+r}.$$

Now we can decompose

$$(1) \quad A = \sum_{\lambda \in \Lambda} A_1.$$

Note that $\Lambda$ is a countable set and let $\Lambda = \{\mu_1, \mu_2, \ldots\}$. (1) means that any $a \in A$ can be written uniquely in the form

$$a = a_1 + a_2 + \cdots + a_l,$$

where $\partial_s(a_i) = \mu_i a_i$, and this infinite sum makes sense in the $m$-adic topology. We have proved:

**Theorem 3.** Let $\partial$ be a $K$-derivation of $A$ such that $\partial(m) \subseteq m$ and $\partial_s \neq 0$. Then, there is a countable semi-group $\Lambda = \{\mu_1, \mu_2, \ldots\}$ contained in $K$ such that any $a \in A$ can be written uniquely in the form

$$a = a_1 + a_2 + \cdots a_l,$$

where $a_i$ satisfies $\partial_s(a_i) = \mu_i a_i$ with $\mu_i \in \Lambda$. Moreover, if we put $A_1 = \{a \in A | \partial_s(a) = \lambda a\}$ for $\lambda \in \Lambda$, we have $A_1 A_r \subseteq A_{1+r}$.

**Example 1.** Let $I$ be an ideal of $R$ generated by homogeneous polynomials.

Then, $\partial = x_1 \partial_1 + x_2 \partial_2 + \cdots + x_n \partial_n$, $x_i = X_i$ mod $I$, is a $K$-derivation of $A$. Clearly, $\partial$ is semi-simple and we have a decomposition

$$A = \sum_{\lambda \in \Lambda} A_1,$$

where $\Lambda = \{a_1 \lambda_1 + a_2 \lambda_2 + \cdots + a_n \lambda_n | a_i \in \Lambda\}$.

$A_1$ is the set of homogeneous elements of degree $n$.

**Example 2.** Let $I$ be an ideal of $R$ generated by monomials. Then, $x_i \partial_i$ is a $K$-derivation of $A$. For any elements $a_1, a_2, \ldots, a_n \in K$, $\partial = a_1 x_1 \partial_1 + a_2 x_2 \partial_2 + \cdots + a_n x_n \partial_n$ is a semi-simple $K$-derivation of $A$. We have

$$A = \sum_{\lambda \in \Lambda} A_1,$$

where $\Lambda = \{k_1 a_1 + k_2 a_2 + \cdots + k_n a_n | k_i \in \mathbb{Z}_{\geq 0}\}$.
Example 3. Let \( A=K[[X,Y,Z]]/(X^2+Y^2+Z^2+XYZ) \). Then,

\[
(X + \frac{1}{2} YZ) \frac{\partial}{\partial Y} - (Y + \frac{1}{2} XZ) \frac{\partial}{\partial X}
\]
induces a \( K \)-derivation \( \partial \) of \( A \). \( \partial_x \) is given by \( \partial_x(x) = -y \), \( \partial_x(y) = x \), where \( x \) and \( y \) are the images of \( X \) and \( Y \) in \( A/m^2 \), respectively. The eigenvalues of \( \partial_x \) are 0, i and \(-i\), where \( i^2 = -1 \). We have

\[
A = \sum_{\lambda \in \Lambda} A_{\lambda},
\]

where \( \Lambda = \{k\imath | k \in \mathbb{Z}\} \).

§3 Exponential of derivation.

In this section, we assume that \( K = \mathbb{C} \), the complex number field. For any \( n \times n \) complex matrix \( A \) we can define the exponential

\[
\exp(A) = \lim_{n \to \infty} (E + A/n)^n = \sum_{k=0}^{\infty} \frac{1}{k!} A^k
\]
of \( A \), see Chapter 1 of [C] for details. If \( AB = BA \) we have \( \exp(A + B) = \exp(A) \exp(B) \). We consider the exponential of a \( \mathbb{C} \)-derivation of a finite dimensional \( \mathbb{C} \)-algebra.

Let \( A \) be a finite dimensional \( \mathbb{C} \)-algebra and \( \partial \) be a \( \mathbb{C} \)-derivation of \( A \). Fix a basis \( x_1, x_2, \ldots, x_n \) of \( A \). Then \( \partial \) is given by a matrix \( D = (a_{ij}) \). We define the exponential \( \exp(\partial) \) to be the linear map given by \( \exp(D) \). The following Lemma 4 and Corollary 5 are easy exercises, cf. Chapter 2 of [V]. Let \( G \) be the \( \mathbb{C} \)-algebra automorphism group of \( A \).

Lemma 4. Let \( A \) be a finite dimensional \( \mathbb{C} \)-algebra. Then a \( \mathbb{C} \)-linear endomorphism \( \partial \) of \( A \) is a \( \mathbb{C} \)-derivation of \( A \) if and only if for any \( t \in \mathbb{R} \), \( \exp(t\partial) \) is a \( \mathbb{C} \)-algebra automorphism of \( A \).

Corollary 5. The Lie algebra of \( G \) is isomorphic to the space of \( \mathbb{C} \)-derivations of \( A \).

We apply these results to a complete local \( \mathbb{C} \)-algebra. Let \( (A, m, \mathbb{C}) \) be a complete local ring of the form \( \mathbb{C}[[X_1, X_2, \ldots, X_n]]/I \). \( G \) denotes the \( \mathbb{C} \)-algebra automorphism group of \( A \) and, for \( k = 1, 2, \ldots, G_k \) the \( \mathbb{C} \)-algebra automorphism group of \( A/m^k \). Note that \( G = \lim_k G_k \). Let \( \text{Der}(A) \) be the set of \( \mathbb{C} \)-derivations of \( A \) and we put \( D_k = \text{Der}(A/m^k) \).

Definition 6. (1) An ideal \( I \) of \( A \) is called a D-ideal if \( \partial(I) \subseteq I \) for all \( \partial \in \text{Der}(A) \).

(2) An ideal \( I \) of \( A \) is called a G-ideal if \( \sigma(I) \subseteq I \) all \( \sigma \in G \). If \( m \) is a D-ideal, we have \( \text{Der}(A) = \lim_k D_k \). Let \( e_k \) be the exponential map \( D_k \to G_k \) given by \( e_k(\partial) = \exp(\partial) \) for \( \partial \in D_k \). Let us assume that \( m \) is a D-ideal and let \( \partial \in \text{Der}(A) \). For \( k = 1, 2, \ldots, \partial \) induces a \( \mathbb{C} \)-derivation \( \partial_k \) of \( A/m^k \). Then,
\[ \sigma = (e_1(\partial_1), e_2(\partial_2), e_3(\partial_3), \cdots) \in \lim_{\kappa} G_k = G. \]

Hence, we can define a map $\text{Exp}: \text{Der}(A) \rightarrow G$. If $\partial, \delta \in \text{Der}(A)$ satisfy $\partial \delta = \delta \partial$ then $\text{Exp}(\partial + \delta) = \text{Exp}(\partial) \text{Exp}(\delta)$. Let $\partial = \partial_h + \partial_n$ be the Jordan decomposition of $\partial$. Then we have $\text{Exp}(\partial) = \text{Exp}(\partial_h) \text{Exp}(\partial_n)$.

**Theorem 7.** If $m$ is a $D$-ideal then a $G$-ideal is a $D$-ideal.

**Proof.** Let $I$ be a $G$-ideal and $\partial \in \text{Der}(A)$. Put $\sigma_t = \text{Exp}(t \partial)$, $t \in \mathbb{R}$. Then, for $k = 1, 2, \cdots$, we have an automorphism $\sigma_{t_k} = \exp(t \partial_k)$ of $A/m^k$. For any $x \in I + m^k/m^k$ we have $\partial_k(x) = \lim_{t \to 0} t^{-1}(\sigma_{t_k}(x) - x) \in I + m^k/m^k$ because $I$ is a $G$-ideal and $A/m^k$ is a finite dimensional $\mathbb{C}$-algebra. This implies that $\partial(I) \subseteq I + m^k$ for $k = 1, 2, \cdots$. It follows that

\[ \partial(I) \subseteq I \cap (I + m_m^k) = I. \]

It is natural to conjecture that Theorem 7 is true for any base field of characteristic 0. If $K = \mathbb{C}$ then $\text{Exp}(\partial_k)$ does not make sense in general but $\text{Exp}(\partial_n)$ does, because $\partial_n$ is nilpotent for all $k$.

We collect several results on $D$-ideals.

**Lemma 8.** Let $K$ be a field with $ch(K) = 0$ and $(A, m, K)$ a complete local $K$-algebra. Then:

1. If $I$ and $J$ are $D$-ideals of $A$ then $\sqrt{I J}, I \cap J, I + J$ and $I : J$ are $D$-ideals.
2. If $I$ is a $D$-ideal then associated primes are $D$-ideals.
3. $A$ is regular if and only if $0$ and $A$ are the only $D$-ideals of $A$.

**Proof.** See [S1] and [S2].

**Proposition 9.** If $A$ is regular then non-zero $G$-ideals are $m$-primary.

**Proof.** Let $I$ a $G$-ideal which is not $m$-primary. We may assume that $I$ is a radical ideal which is not maximal. Let $a \in m^2$ be an element which is not a zero-divisor of $A/I$ and $\partial$ be a $C$-derivation of $A$. Then $a \partial$ induces a nilpotent derivation of $A/m^k$, for $k = 1, 2, \cdots$. Hence $\text{Exp}(a \partial)$ is a well defined automorphism of $A$. For any $x \in I$, we have $\text{Exp}(a \partial)(x) - x \in aA \cap I = aI$ and $a^{-1}(\text{Exp}(a \partial)(x) - x) \in I$. But, we have $a^{-1}(\text{Exp}(a \partial)(x) - x) = \partial(x) + ab$ for some $b \in A$. Therefore, we have $\partial(I) \subseteq I + aA$. Replacing $a$ by $a^k$, we have $\partial(I) \subseteq I + a^kA$ for $k = 1, 2, \cdots$. Hence we have $\partial(I) \subseteq I$, $I$ is a $D$-ideal and, by Lemma 8, $I = 0$.

**Theorem 10.** Let $(A, m, \mathbb{C})$ be a complete local $\mathbb{C}$-algebra. If $A$ is not regular and $m$ in a $D$-ideal then, for $p \in \text{Spec} A$, $A_p$ is not regular if $p$ contains a $G$-ideal $I$ such that $ht(I) > 0$. 
**Proof.** It is enough to show that \( V(I) \) is contained the singular locus \( \text{Sing}(A) \) of \( A \). Let \( J \) be the defining ideal of \( \text{Sing}(A) \). Suppose that there is a minimal prime \( p \) of \( I \) such that \( p \supseteq J \). By Lemma 8 and Theorem 6, \( p \) is a \( D \)-ideal. We can find a system of parameters \( x_1, x_2, \ldots, x_d \) of \( A \) so that \( x_1, x_2, \ldots, x_r \) form a system of parameters of \( A/p \). Then, \( \mathbb{C}[[x_1, x_2, \ldots, x_r]] \cap p = 0 \) and \( A/p \) is finite over the image of \( \mathbb{C}[[x_1, x_2, \ldots, x_r]] \) in \( A/p \). Let \( S = \mathbb{C}[[x_1, x_2, \ldots, x_r]] \) and \( L \) the field of fractions of \( S \). Then the field of fractions \( M \) of \( A/p \) is a finite separable extension of \( L \). We apply the theory of universally finite module of differentials to our complete local rings. See [SS] for fundamental theory of universally finite module of differentials, see also [K]. Let \( \Omega_A \) be the universally finite module of differentials of \( A \) over \( \mathbb{C} \). Then \( \text{Der}(A) \) is isomorphic to \( \text{Hom}_\mathbb{A}(\Omega_A, A) \). Since \( \Omega_A \) is a finite \( A \)-module and \( A_p \) is flat over \( A \), we have \( \text{Der}(A_p) = \text{Hom}_\mathbb{A}(\Omega_A \otimes_\mathbb{A} A_p, A_p) = \text{Der}(A) \otimes_\mathbb{A} A_p \). Then we have \( \mathfrak{a}_p (A_p) \subseteq pA_p \) for all \( \mathfrak{a} \in \text{Der}(A_p) \). Let \( B \) be the \( pA_p \)-adic completion of \( A_p \). By [SS], the universally finite module of differentials of \( B \) over \( \mathbb{C} \) is isomorphic to \( \Omega_A \otimes_\mathbb{A} B \) and hence, we have \( \text{Der}(B) = \text{Der}(A_p) \otimes_\mathbb{A} B \). From the exact sequence

\[
\Omega_B \otimes_\mathbb{A} A \rightarrow \Omega_A \rightarrow \Omega_{A/S} \rightarrow 0,
\]

we have \( \text{Der}_A(B) = \text{Hom}_\mathbb{A}(\Omega_{A/S}, B) \subseteq \text{Der}(B) \). Since \( M \) is finite separable over \( L \) we have \( \text{Der}_M(B) = \text{Der}_L(B) \). We see that \( \text{Der}_M(B) \) is a submodule of \( \text{Der}(B) = \text{Der}(A) \otimes_\mathbb{A} B \). Hence, \( pB \) is a \( D \)-ideal of \( B = M[[Y_1, Y_2, \ldots, Y_r]] \). By Lemma 8, we get \( pB = 0 \) and consequently, \( pA_p = 0 \), but this contradicts to \( \text{ht}(p) > 0 \). Therefore, every minimal prime of \( I \) contains the defining ideal of \( \text{Sing}(A) \).

**References**


